A Study of Geometry, Physics and Integrability of Geodesics on Curved Spaces

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Chapter 1

Introduction

Mechanics is an ancient subject that has been studied by civilizations throughout history all around the world, with many applications intended in physics and mostly engineering. From the beginning of my career in physics, I was fascinated by the study of motion of physical objects in space-time, ie. the dynamical perspective of mechanics. Its various forms include classical mechanics, and relativistic mechanics which have been discussed in this thesis. Classical mechanics can be said to derive from a more general theory of physics, known as relativistic mechanics, which studies mechanics from a more geometric point of view in both special and general versions. In all cases, mechanics revolves around trajectories, which according to Hamilton's principle of least action, define the path of least action between any two chosen points on the particle's journey. Such trajectories are geometrically known as geodesics.

A geodesic defines a trajectory with the shortest distance geometrically between any two chosen points that mark the start and end of a particle's trajectory, the study of which has received significant attention in mechanics in all its forms. From it, the Euler Lagrange equations are derived that produce all the equations of motion for a given system. Under certain circumstances, such as a symmetry available due to a cyclical variable, we can modify geodesics to obtain a simpler picture of the trajectory of the particle. Furthermore, in the topic of Relativity, the null geodesic is specially relevant because the speed of traversal along such trajectories remains unchanged in all inertial frames, which is reflected in Einstein's postulates.

The subject of Integrable systems deals with non-linear differential equations that ideally are analytically solvable. This implies reducibility of the solution to a finite number of algebraic operations and integrations. Realistically, it is difficult to find such systems that are exactly solvable into explicit solutions. Such systems have been studied as early as in the fundamental works of Euler, Liouville, Riemann, Poincaré, and others. Here, wherever we discuss analysis of the integrability of the system, we merely imply the solvability of the available differential equations. One example of integrable systems in classical mechanics is the topic of action-angle variables.

While Euclidean spaces are not directly used to describe space-times or gravity, comparison of solutions on such spaces that are similar to problems on Minkowski spaces under a Wick's rotation, can help provide solutions. They are useful playgrounds for studying self-dual mechanical systems, some of which are called instantons. Self-dual metrics with Euclidean signature describe gravitational instantons, which comprise a subset of Yang-Mills instantons that derive from self-dual Yang-Mills (SDYM) gauge fields. The significance of

SDYM in integrable systems arises from R.S. Ward's conjecture that perhaps all solvable or integrable partial differential equations are merely various reductions of the SDYM equations. Thus, the SDYM system is a generator of integrable systems that provides a general geometric foundation for their analysis. They have numerous applications in mathematics and physics, appearing in gauge theory, classical general relativity, and the analysis of 4-manifolds.

One such reduction produces the general Darboux-Halphen system, of which a special case has applications in mathematical physics in relation to the study of magnetic monopole dynamics, self-dual Einstein equations, and topological field theory. One example of great interest is the Taub-NUT space-time, discussed in this thesis in great detail.

Gravity has been studied, ever since its discovery as a force, to describe planetary motion in the subject of astronomy. Johannes Kepler prescribed three laws describing planetary motion between 1609 and 1619, improving Nicolaus Copernicus' heliocentric theory, using elliptic orbits instead of circular ones. Isaac Newton further showed in 1687 that Kepler's laws under a good approximation was equivalent to the combination of his three laws of motion and his law of universal gravitation, which defines the dependence of gravitational force on mass and distance via the inverse square law. Upon presentation of Newton's work in his book *Principia*, in 1686, Robert Hooke claimed that Newton had obtained the inverse square law from him.

The inverse square law governing the magnitude over distance was formulated by Newton, analogous to the Coloumb force law in electrostatics. While it may exhibit laws similar to those of electrostatics and magnetism, such as the inverse square force law, and the Poisson's equation, gravity can be supposed to be a consequence of geometry. This is because the formula for acceleration lacks any physical quantities, unlike its equivalent in electrostatics, which involves charge and mass terms. This makes gravity more universal, since all objects, massive or not (massless like photons) will abide by this force law. Interestingly, in classical mechanics, according to Bertrand's theorem, a mechanical system governed by the inverse square law force is conformally dual to a mechanical system driven by Hooke's law force for harmonic oscillators.

General Theory of Relativity pioneered and published by Albert Einstein in 1915, a geometric theory of gravitation, has superseded Newton's law of gravitation, and is an indispensable part of modern physics, such as astrophysics, cosmology, string theory and particle physics. It also has practical uses in Global Positioning System. It tries to describe gravity not as a physical force, but as a consequence of the curvature of space-time. This curvature can be defined as a perturbation of flat space-time, resulting in a deformation of geodesics from straight lines for flat spaces. Furthermore, the geodesic equation in General Relativity is a non-linear differential equation that is not always directly solvable, a topic elaborated in the subject of Integrable Systems.

My thesis has been organized as follows:

We start with the study of reductions of geodesics via projection onto a hypersurface characterised by a conserved quantity. The formulation of this reduction theory, known as the Jacobi-Maupertuis theory, was described only for autonomous mechanical systems. Its relativistic version was shown consistent with the classical version under non-relativistic approximations, and the formulation was extended to time-dependent systems. The autonomous system formulation was applied to the Kepler and Liénard mechanical systems, and analyzed.

After that, we undergo an exercise during my foray into the study of instantons, involving exploration of the Schwarzschild instanton. This exercise starts by considering the Euclidean Schwarzschild metric, and analyzing it geometrically and topologically.

Then I proceeded to study the mechanics of self-dual curved spaces starting with the study of self-dual Bianchi-IX metric and the related systems. This study extends to equations that derive from the reduction of SDYM equation, finally followed by a detailed geometric and dynamical analysis of the Taub-NUT system, one special case of the self-dual Bianchi-IX system.

The final area of my work focuses on generalising the formulation of relativistic mechanics from particle on flat space to particle on curved space due a single gravitational potential, and comparison with a pre-existing ad-hoc formulation. This formulation starts with a metric where the gravitational potential acts as a perturbation of flat space. A modified local Lorentz transformation is also defined on such spaces.

Chapter 2

Jacobi-Maupertuis metric as reduction of geodesic flow

Authors: S. Chanda, G.W. Gibbons and P. Guha.

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2.1 Introduction

Riemann studied concepts like curvature and geodesics by introducing Riemannian manifolds in his Habilitationsthesis, where he defined an inner product on every tangent space of a manifold. Such inner products were defined via a structure known as the metric that defines infinitesimal length elements locally on the tangent space, which can be integrated to compute a given path's length [1, 2] between any two points on the manifold. The shortest path in terms of integrated path length is defined as the geodesic, which according to Maupertuis, is effectively the path of least action, comparable to Fermat's path of least time for light [3]. On these manifolds, the form of the action integrals along geodesics is known as the Maupertuis form of action [4] along geodesics, about which the integrand is an exact differential. In this chapter we will focus on geodesics and their projection onto the constant energy hypersurface.

The Jacobi-Maupertius (JM) metric is a conformal projection of the a space-time action functional onto a fixed energy spatial hypersurface, reducing the problem to a spatial geodesic [1], whose momentum is confined to a unit momentum sphere. In other words, the Jacobi-Maupertuis metric reformulates Newton's equations as geodesic equations for a Riemannian metric which degenerate at the Hill boundary [5]. The Jacobi metric formulation is a procedure for producing a geodesic from a given Hamiltonian. Such trajectories of a Hamiltonian system can be viewed as geodesics of a corresponding configuration space or its enlargement under some constraints. Since we parameterize with respect to time $\tau = t$, the term quadratic in time is present as the potential.

In the theory of integrable systems, each available first-integral or conserved quantity reduces the degrees of freedom of the mechanical problem one has to deal with by one. Similarly in the Jacobi-Maupertuis theory, the conserved Hamiltonian enables the conversion of the classical action into a geodesic of reduced dimensionality. Since such Hamiltonians often already arise from Lagrangians originating from a metric, the Jacobi metric formulation is a reduction of a geodesic from higher dimensions to lower dimensions.

Recently, it was shown [6] that free motion of massive particles in static space-times is given by geodesics of an energy-dependent Riemannian metric on the spatial sections analogous to Jacobi's metric in classical dynamics. Recently this result has been extended [7] to explore the Jacobi metrics for various stationary metrics. In particular, the Jacobi-Maupertuis metric is formulated for time-dependent metrics by including the Eisenhart-Duval lift, known as the Jacobi-Eisenhart metric. An important application to gravity was shown [8] by Ong who studied the curvature of the the Jacobi metric for the Newtonian n-body problem. For $n = 2$, the problem reduces to the Kepler's problem of the relative motion and the relevant Jacobi metric is up to an unimportant overall constant factor.

The Kepler system, derived by Johannes Kepler in 1609, as interpreted by Newton, is a 3-dimensional integrable system for an inverse square law force describing elliptic trajectories [9, 10]. It is related to the isotropic oscillator system via a canonical type transformation known as the Bohlin transformation, resulting in many properties of the two systems being inter-related. It has many integrals of motion such as the angular momentum, the Hamiltonian and the Runge-Lenz vector. The last two translate into the equivalent conserved quantities known as the Fradkin tensors for the oscillator system under Bohlin's transformation. Recently Kepler problem has been studied on noncommutative κ -spacetime and corresponding Bohlin-Arnold duality [11]. In particular, regularization of the Kepler problem on κ -spacetime in several different ways [12]. Regularization is a mathematical procedure to cure this singularity. A nice clear treatment of regularizing the Kepler problem was done by Moser in his 1970 paper [13], the treatment of Moser relates the Kepler flow for a fixed negative energy level to the geodesic flow on the sphere S^n. A lucid analysis of the geometrical aspects of Kepler problem can be found in Milnor [14]. Belbruno extended the cases of positive energy to negative energy, in correspondence to the 3-hyperboloid $\mathcal{H}^3$, and zero energy which corresponds to 3-dimensional Euclidean space [15].

In 1941, G. Randers [19] introduced a Finsler metric by modifying a Riemannian metric $g = g_{ij}\, dx^i \otimes dx^j$ by a linear term $b = b_i(x)dx^i$, the resulting norm on the tangent space is given by

$$F(x, y) = \sqrt{g_{ij}y^i y^j} + b_i(x)y^i, \qquad y = y^i \partial_{x^i} \in T_x M.$$

Randers metrics have received much attention [20, 21] lately because these yield the solutions to Zermelo's problem of navigation, most recently, it has been extended to quantum navigation problem of finding the time-optimal control Hamiltonian [22]. In [23], E. Zermelo studied a classical control problem to find a deviation of geodesics under the action of a time-dependent vector field.

In the Zermelo construction, we can describe a given stationary metric g_{ij} in terms of a vector field or drift (*wind*) W^i, and a another different metric h_{ij}, describing Zermelo data $\{h_{ij}, W^i\}$ [24]. For a time-independent wind, Shen [25] showed that trajectories of least travel time are particular Finsler geometry geodesics known as Rander's metric. This model physically describes fluid dynamical analogue models of a rotating black hole akin to a test particle drifting with a spinning fluid vortex. A stationary space-time rewritten into Zermelo form will partially involve Painlevé-Gullstrand form [26, 27].

In this chapter, in section 2.2 we will first explore three different, but equivalent approaches to obtaining the Jacobi metric. In the first one, we start with the regular formulation of the action with the Lagrangian for autonomous mechanical systems. We shall cover two ways of formulating the Jacobi metric with this approach: by equating the action to a Lorentz invariant line element integral, and by redefining the system from a constant energy hypersurface to a unit momentum hypersurface, where the kinetic energy is rescaled by a conformal factor to unity [1, 16].

We then proceed in section 2.3 to obtain the Jacobi metric purely from the line element integral of Rander's form of stationary metric, essentially reproducing the formulation used in [17], while the author in [17] employed a static metric and a Zermelo form for the stationary metric [18]. Here, we will go a step further, and apply the non-relativistic approximation to this result, thereby reproducing the previous result and equating the two formulations. Afterwards, in section 2.4, we apply the formulation to various examples such as Schwarzschild, Taub-NUT, Bertrand and Kerr spacetimes.

In section 2.5 we shall next obtain the Jacobi metric for time-dependent mechanical systems. So far, the formulation has only been applied to time-independent static and stationary metrics. The difficulty in application to time-dependent metrics is the absence of a constant energy hypersurface. To resolve this issue, we modify the metric via the Eisenhart-Duval lift introduced by Eisenhart [28] and rediscovered by Duval et. al. [29]. This means introducing an extra dimension via a dummy variable and a fixed hypersurface on which to project the geodesic, thus relating n dimensional mechanics to geodesics on $n+2$ dimensional space. First, we demonstrate the utility of the Eisenhart-Duval lift in this context, by describing autonomous and non-autonomous systems with and without the lift applied, then deduce the formulation from the line-element integral, and finally apply limits for a non-relativistic approximation. We propose calling the result the Jacobi-Eisenhart metric. Finally, in section 2.6, we obtain the same results using projective transformations and compare them to verify consistency of the results.

Finally, in section 2.7 the Kepler system will be shown to be geodesic flow on constant curvature surfaces. Here, we shall demonstrate how such a projection to a fixed energy surface following a canonical transformation is the Bohlin's transformation [30] that converts the oscillator system into the Kepler system. This will be followed by a discussion on application in Houri's canonical transformation [31]. First we shall couple it with Milnor's construction to study the preservation of the form of geodesic flows under such canonical transformations.

2.2 Jacobi-Maupertuis Theory: Preliminaries

Let g be a Riemannian metric on the manifold M. If $\dot{x} \in T_xM$, then its length is

$$||\dot{x}|| := \sqrt{g_x(\dot{x}, \dot{x})}.$$

If $\gamma : [a, b] \to M$ is a smooth curve in M, then $\dfrac{d\gamma}{d\tau} \in T_{\gamma(\tau)}M$, which allows us to define the length of the curve γ [1, 2] as

$$l(\gamma) := \int_a^b \left|\left|\frac{d\gamma(\tau)}{d\tau}\right|\right|_{\gamma(t)} d\tau.$$

where for the geodesic, the following condition holds:

$$\delta\ l(\gamma) = 0.$$

The geodesic can also be defined as follows:

Definition 2.2.1. A geodesic in a pseudo-Riemannian manifold (M, g) is a solution to the Euler-Lagrange equations

$$[\mathcal{L}]^x := \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i}\right) - \frac{\partial \mathcal{L}}{\partial x^i} = 0. \tag{2.2.1}$$

where the Lagrangian $\mathcal{L} : TM \to \mathbb{R}$ is defined by $\mathcal{L} = \frac{1}{2} g_x(\dot{x}; \dot{x})$.

Consider an action integral I_{12} along a path parameterized by τ between two points 1 and 2, defined by

$$I_{12} = \int_1^2 dI = \int_1^2 d\tau \; L. \tag{2.2.2}$$

where $L = L(\boldsymbol{x}, \dot{\boldsymbol{x}})$ is a function quadratic in velocity, the position being $\boldsymbol{x}$ and the velocity being $\dot{\boldsymbol{x}}$. The geodesic is characterised by the Euler-Lagrange equation (2.2.1) which is derivable from

$$\delta I_{12} = \int_1^2 \delta(dI) = \int_1^2 d\tau \; \delta L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = 0$$

This means that if we vary the line integral (2.2.2) and apply (2.2.1), we have

$$\delta I_{12} = \int_1^2 d\tau \left(\frac{\partial L}{\partial x^i}\delta x^i + \frac{\partial L}{\partial \dot{x}^i}\delta \dot{x}^i\right) = \int_1^2 d\tau \left[\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\delta x^i + \frac{\partial L}{\partial \dot{x}^i}\frac{d}{d\tau}\left(\delta x^i\right)\right] = \int_1^2 d\tau \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^i}\delta x^i\right),$$

$$\therefore \quad \int_1^2 \delta(dI) = \int_1^2 d\left(\frac{\partial L}{\partial \dot{x}^i}\delta x^i\right).$$

Since we are considering the path of extremal variation, we are dealing with an integral that is locally exact about the geodesic, (ie. $\delta(dI) = d(\delta I)$). This means on substituting the momentum $p_i = \dfrac{\partial L}{\partial \dot{x}^i}$, the effective integral along the geodesic and the effective Lagrangian L_{geod} along the geodesic are given by

$$\int_1^2 d(\delta I) = \int_1^2 d\left(\frac{\partial L}{\partial \dot{x}^i}\delta x^i\right) \qquad \Rightarrow \qquad \delta I = \frac{\partial L}{\partial \dot{x}^i}\delta x^i \quad \Rightarrow \quad dI = \frac{\partial L}{\partial \dot{x}^i} dx^i,$$

$$\Rightarrow \qquad I_{12} = \int_1^2 dI = \int_1^2 \frac{\partial L}{\partial \dot{x}^i} dx^i = \int_1^2 d\tau \left(\frac{\partial L}{\partial \dot{x}^i}\dot{x}^i\right) = \int_1^2 p_i dx^i,$$

$$\therefore \qquad L_{geod} = \frac{\partial L}{\partial \dot{x}^i}\dot{x}^i = p_i \dot{x}^i. \tag{2.2.3}$$

where the effective line integral is known as the Maupertuis form [4] of the line integral.

2.2.1 Classical Jacobi metric from Natural Hamiltonian

Mechanics has been historically studied from two approaches: Lagrange's and Hamilton's. This results in two different, yet equivalent formulations of the equations of motion to describe geodesics. Since we have shown how to formulate the lifted Hamiltonian and Lagrangian, it is natural to explore how the equations of motion take shape under such formulations, and the effect on conserved quantities.

If one starts with a static metric ($g_{0i} = 0$) on a given $n+1$ dimensional space-time

$$dl^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{00}\ dt^2 + g_{ij}\ dx^i dx^j.$$

it is a simple matter to formulate the corresponding Lagrangian describing the dynamics on that space. Such dynamical systems under affine parametrization $\tau = x^0 = t$ are defined by the mechanical action and its related Lagrangian:

$$S = \int_{\tau_1}^{\tau_2} d\tau\ L(\boldsymbol{x}, \dot{\boldsymbol{x}}), \tag{2.2.1.1}$$

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \frac{m}{2} g_{\mu\nu}(\boldsymbol{x}) \dot{x}^\mu \dot{x}^\nu = \frac{m}{2} g_{ij} \dot{x}^i \dot{x}^j - U(\boldsymbol{x}) \equiv T - U(\boldsymbol{x}), \tag{2.2.1.2}$$

and the Euler-Lagrange equation is given by:

$$\ddot{x}^i = -\sum_{jk} \Gamma^i_{jk} \dot{x}^j \dot{x}^k - \sum_l g^{il}(x) \partial_l U(x)\,. \tag{2.2.1.3}$$

If the Lagrangian can have a natural form given by (2.2.1.2), then so will the Hamiltonian when momentum has been solved for velocity and substitute back in the Hamiltonian. The natural Hamiltonian for a time-independent dynamical system that acts as the generator for time-translations is a conserved quantity is given by a Legendre transformation

$$H(\boldsymbol{x}, \boldsymbol{p}) = \sum_{i=1}^{n} p_i \dot{x}^i - L(\boldsymbol{x}, \dot{\boldsymbol{x}}) \qquad\qquad p_i = \frac{\partial L}{\partial x^i} = g_{ij}(\boldsymbol{x}) \dot{x}^j.$$

$$H(\boldsymbol{x}, \boldsymbol{p}) = \frac{1}{2m} g^{ij}(\boldsymbol{x}) p_i p_j + U(\boldsymbol{x}) \equiv T(\boldsymbol{x}, \dot{\boldsymbol{x}}) + U(\boldsymbol{x}) = E. \tag{2.2.1.4}$$

where the dynamical equations or Hamilton's equations of motion are:

$$\begin{aligned} \dot{x}^i &= \frac{\partial H}{\partial p_i} = \frac{g^{ij}(\boldsymbol{x})}{m} p_j, \\ \dot{p}_i &= \frac{\partial H}{\partial x^i} = \frac{1}{2m} \frac{\partial g^{ij}(\boldsymbol{x})}{\partial x^i} p_i p_j + \frac{\partial U}{\partial x^i}. \end{aligned} \tag{2.2.1.5}$$

This means that the Lagrangian of (2.2.1.2) can be written as

$$L = 2T - E.$$

and thus the zero-variation equation of the action (2.2.1.1) can be written as

$$\delta S = \delta\Big(\int_{\tau_1}^{\tau^2} d\tau\ L \Big) = \delta \left[\int_{\tau_1}^{\tau^2} d\tau\, (2T - E) \right] = 2 \int_{\tau_1}^{\tau^2} d\tau\ \delta T.$$

Thus, the effective action is given as:

$$S_{eff} = \int_{\tau_1}^{\tau_2} d\tau\ 2T. \tag{2.2.1.6}$$

Being the generator of time translations, the time derivative of any functions is given by Poisson Bracket operations $\dot{f} = \{f, H\}$. Naturally, any conserved quantities will be in involution with this Hamiltonian, itself being a conserved quantity:

$$\dot{Q} = \{Q, H\} = 0.$$

This Hamiltonian is made up of 2 parts; quadratic and potential. In the next section, we shall see how to reduce it to being homogeneously quadratic.

From (2.2.1.6), we can see that for conserved quantities, an alternative formula for the action will suffice to describe geodesics with conserved energies. This effective Lagrangian based action integral may be equated to a metric line-element integral [4] using (2.2.1.4) as follows:

$$S_{eff} = \int_{\tau_1}^{\tau_2} d\tau\ 2T = \int_{\tau_1}^{\tau_2} d\tau\ \sqrt{2T}\sqrt{2T} = \int_{\tau_1}^{\tau_2} d\tau\ \sqrt{2(E-U)}\sqrt{m g_{ij}(\boldsymbol{x})\dot{x}^i\dot{x}^j},$$

$$S_{eff} = \int_{\tau_1}^{\tau_2} d\tau\ \sqrt{2m(E-U)g_{ij}(\boldsymbol{x})\dot{x}^i\dot{x}^j} \equiv \int_1^2 d\tau \sqrt{\left(\frac{dl_{eff}}{d\tau}\right)^2}.$$

Thus, the effective Jacobi metric can be given as

$$dl_{eff}^2 = Ldt^2 = 4\left(E - U(\boldsymbol{x})\right)Tdt^2 \qquad T = \frac{m}{2}g_{ij}(\boldsymbol{x})\dot{x}^i\dot{x}^j,$$

$$\Rightarrow \qquad dl_{eff}^2 = 2m\left(E-U\right)g_{ij}(\boldsymbol{x})dx^i dx^j.$$

We can view the solution curves of natural mechanical systems as the geodesics of a special metric. This process allows us to convert the Hamiltonian $n+1$ dimensional system into a spatial n-dimensional geodesic with a rescaled conserved Hamiltonian:

$$g^{ij}(\boldsymbol{x})p_i p_j = 2m\left(E - U(\boldsymbol{x})\right) \qquad \Rightarrow \qquad \widetilde{H} = \frac{g^{ij}(\boldsymbol{x})}{2m\left(E-U(\boldsymbol{x})\right)}p_i p_j = 1. \tag{2.2.1.7}$$

We have essentially taken the kinetic energy part of the total conserved energy of the system, and rescaled it with a conformal factor that is its inverse into an equivalent constrained system with unit momentum sphere. This means that the metric and its inverse transform into the Jacobi metric as follows:

$$\begin{gathered}\widetilde{g}^{ij}(\boldsymbol{x})p_i p_j = 1, \\ \widetilde{g}^{ij}(\boldsymbol{x}) = \frac{g^{ij}(\boldsymbol{x})}{2m\left[E-U(\boldsymbol{x})\right]} \qquad \Rightarrow \qquad \widetilde{g}_{ij}(\boldsymbol{x}) = 2m\left[E - U(\boldsymbol{x})\right]g_{ij}(\boldsymbol{x}).\end{gathered} \tag{2.2.1.8}$$

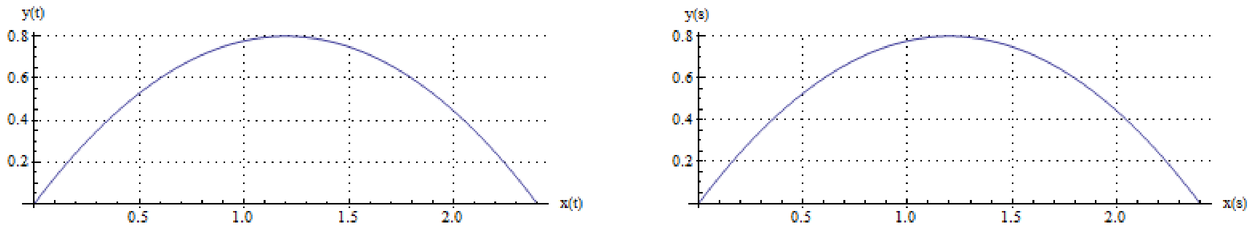
where the kinetic energy part of the system serves as the conformal factor. We can summarize the details with the following theorem.

Theorem 2.2.1 (Jacobi-Maupertuis principle)**.** *Let $T : TM \to \mathbb{R}$ be a smooth pseudo-Riemannian metric and let $U : M \to \mathbb{R}$ be a smooth potential energy function. Let $t \mapsto x(t), I \to M$ be a curve in M such that $H\left(x(t), \frac{dx(t)}{dt}\right) = E \in \mathbb{R}$ and $U(x(t)) \neq E$ for all t, and $\sigma(t) \mapsto x(\sigma), J \to M$ be another curve in M such that $H\left(x(\sigma), \frac{dx(\sigma)}{d\sigma}\right) = 1$ for all σ. Then the map $t \mapsto \sigma(t), I \to \mathbb{R}$ defined by*

$$\sigma(t) = 2m\int_0^t dt\left[E - U(x(t))\right].$$

is a diffeomorphism onto its image J. We denote its inverse by $\sigma \mapsto t(\sigma); J \to I$. Moreover, the curve $t \mapsto x(t)$ in M is a solution to the Euler-Lagrange equation $[T-U]^x = 0$ (see 2.2.1), iff the curve $\sigma \mapsto x(t(\sigma)), J \to M$ is a geodesic of the "Jacobi metric"

$$\widetilde{T} = 2m\left(E-U\right)T.$$



Trajectory for projectile

Lagrangian: $L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - 10y.$

Equations of motion: $\ddot{x} = 0, \qquad \ddot{y} = -10.$

Initial conditions: $x(0) = y(0) = 0,$
$\dot{x}(0) = 3ms^{-1}, \quad \dot{y}(0) = 4ms^{-1}.$

Geodesic of projectile's Jacobi metric

Jacobi metric: $ds^2 = (25 - 20y)\left(dx^2 + dy^2\right).$

Equations of motion:

$$x'' = \frac{10}{12.5 - 10y}\, x'y',$$
$$y'' = \frac{5}{12.5 - 10y}\left[(y')^2 - (x')^2\right].$$

Initial conditions: $x(0) = y(0) = 0,$
$x'(0) = \frac{3}{25}, \quad y'(0) = \frac{4}{25}.$

Figure 2.1: Example of Jacobi metric for projectile in Earth's gravity

Proof. So long as we have

$$\frac{d\sigma(t)}{dt} = 2m\left[E - U(x(t))\right] \neq 0$$

the inverse function theorem guarantees that $t \mapsto s(t)$ is a diffeomorphism onto its image M, reparameterizing the the curve as $s \mapsto x(s) = x(t(s))$. Thus, the velocity upon differentiation wrt t is:

$$\frac{dx^i}{dt} = \frac{dx^i}{d\sigma}\frac{d\sigma}{dt} = (E - U(x))\,\frac{dx^i}{d\sigma} \tag{2.2.1.9}$$

and the acceleration from (2.2.1.3) can be re-written as:

$$\ddot{x}^i = \frac{d\sigma}{dt}\frac{d}{d\sigma}\left(\frac{d\sigma}{dt}\frac{dx^i}{d\sigma}\right) = (E - U(x))^2\,\frac{d^2x^i}{d\sigma^2} - (E - U(x))\,\partial_j U(x)\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma} \tag{2.2.1.10}$$

and the Euler-Lagrange equation (2.2.1.3) transforms as:

$$(E - U(x))\,\frac{d^2x^i}{d\sigma^2} - \partial_j U(x)\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma} = -\sum_{jk}(E - U(x))\,\Gamma^i_{jk}\frac{dx^j}{d\sigma}\frac{dx^k}{d\sigma} - \sum_l \widetilde{g}^{il}\partial_l U(x)$$

$$\Gamma^i_{jk} = \left[\frac{1}{2\,(E - U(x))}\left(\partial_j U(x)\delta^i_k + \partial_k U(x)\delta^i_j - \widetilde{g}^{im}\partial_m U(x)\widetilde{g}_{jk}\right) + \widetilde{\Gamma}^i_{jk}\right],$$

$$(E - U(x))\,\frac{d^2x^i}{d\sigma^2} - \partial_l U(x)\frac{dx^l}{d\sigma}\frac{dx^i}{d\sigma} = -\sum_{jkl}\left[(E - U(x))\,\Gamma^i_{jk}\frac{dx^j}{d\sigma}\frac{dx^k}{d\sigma} + \widetilde{g}^{il}\partial_l U(x)\right]$$
$$= -\sum_{jkl}\left[\partial_l U(x)\frac{dx^l}{d\sigma}\frac{dx^i}{d\sigma} - \widetilde{g}^{im}\partial_m U(x)\left(\frac{1}{2}\widetilde{g}_{jk}\frac{dx^j}{d\sigma}\frac{dx^k}{d\sigma}\right) + \widetilde{g}^{il}\partial_l U(x)\right] - \sum_{jk}(E - U(x))\,\widetilde{\Gamma}^i_{jk}\frac{dx^j}{d\sigma}\frac{dx^k}{d\sigma}$$

$$\frac{d^2x^i}{d\sigma^2} = -\widetilde{\Gamma}^i_{jk}\frac{dx^j}{d\sigma}\frac{dx^k}{d\sigma} \tag{2.2.1.11}$$

Thus, the Euler-Lagrange equation has been mapped to a regular geodesic equation for the Jacobi metric (2.2.1.8). The Jacobi-Maupertius principle holds for any system with non-zero kinetic energy. □

One may ask why according to (2.2.1.4) we are not substituting $T(\boldsymbol{x}, \dot{\boldsymbol{x}}) = E - U(\boldsymbol{x})$. The reason is that doing so in (2.2.1.8) would effectively make the metric tensor components velocity dependent

$$\widetilde{g}_{ij}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = 2mT(\boldsymbol{x}, \dot{\boldsymbol{x}})g_{ij}(\boldsymbol{x}).$$

like the Finsler metric. Naturally, for a transformed Hamiltonian, the dynamical description should also change.

2.2.2 Extended Hamiltonian formulation

The Maupertuis transformation $X_H \to \widetilde{X}_{\widetilde{H}}$ relates two vector field on M. If t and σ are time along trajectories of the vector fields X_H and $\widetilde{X}_{\widetilde{H}}$, then

$$d\sigma = (E - U(x))dt \tag{2.2.2.1}$$

The distinguished role of the time t is not desirable in the general case of non-autonomous Hamiltonian systems. We therefore introduce an evolution parameter s to parameterize time evolution of the system. In the extended formalism, time t is treated as an ordinary canonical function $t(s) \equiv x^0(s)$ of an evolution parameter s. We may conceive a 'new' momentum coordinate $p_0(s)$ in conjunction with the time as an additional pair of canonically conjugate coordinates. The extended Hamiltonian $\mathcal{H}(x^0, p_0, x^i, p_i)$ is defined as a differentiable function on the cotangent bundle $T^*Q = T^*(\mathbb{R} \times M)$ endowed with a chart $(p_0, p_i) \in T^*_{x_0, x_i}Q$ with $\frac{\partial \mathcal{H}}{\partial s} = 0$. It is given by $\mathcal{H}(x^0, p_0, x^i, p_i) = H(x^i, p_i, x^0) + p_0$, where x^0 and p_0 are conjugate variables and $p_0 = -H$. The extended phase space admits a Liouville form (or integral invariant of Poincaré-Cartan)

$$\theta_{\mathcal{H}} = p_0 dt + p_i dx^i \tag{2.2.2.2}$$

and the Hamiltonian flow is completely determined by the conditions:

$$\langle \mathbb{X}_{\mathcal{H}}, dt \rangle = 1 \quad \text{and} \qquad \mathbb{X}_{\mathcal{H}} \lrcorner d\theta_{\mathcal{H}} = 0, \qquad\qquad \text{where} \quad \mathbb{X}_{\mathcal{H}} = \dot{x}^\mu \frac{\partial}{\partial x^\mu} + \dot{p}_\mu \frac{\partial}{\partial p_\mu}. \tag{2.2.2.3}$$

Invoking Hamilton's equations of motion, and keeping in mind that $\dot{t} = 1, p_0 = -q(t)$ and the Maupertuis form of action according to (2.2.3), we have

$$\mathcal{L}(x^\mu, \dot{x}^\mu) = \sum_{\mu=0}^{n} p_\mu \dot{x}^\mu = \sum_{i=1}^{n} p_i \dot{x}^i + p_0 \dot{t}. \tag{2.2.2.4}$$

we have the extended Hamiltonian given below:

$$\mathcal{H}(x^i, p_i, t) = \sum_{\mu=0}^{n} p_\mu \dot{x}^\mu - \mathcal{L}(x^\mu, \dot{x}^\mu) = \left[\sum_{i=1}^{n} p_i \dot{x}^i - \mathcal{L}(x^i, \dot{x}^i, t)\right] + p_0 \dot{t} = 0,$$

$$\mathcal{H}(x^i, p_i, t) = H(x^i, p_i, t) - q(t) = 0.$$

Thus, the extended Hamiltonian vector field is given by

$$\mathbb{X}_{\mathcal{H}} = \sum_{\mu}\left(\frac{\partial\mathcal{H}}{\partial p_\mu}\frac{\partial}{\partial x^\mu} - \frac{\partial\mathcal{H}}{\partial x^\mu}\frac{\partial}{\partial p_\mu}\right) \quad = \quad \sum_{i}\left(\frac{\partial\mathcal{H}}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial\mathcal{H}}{\partial x^i}\frac{\partial}{\partial p_i}\right) + \frac{\partial\mathcal{H}}{\partial H}\frac{\partial}{\partial t} - \frac{\partial\mathcal{H}}{\partial t}\frac{\partial}{\partial H}$$

Here, we apply some rules:

$$\frac{\partial\mathcal{H}}{\partial x^i} = \frac{\partial H}{\partial x^i}, \qquad \frac{\partial\mathcal{H}}{\partial p_i} = \frac{\partial H}{\partial p_i}, \qquad \frac{\partial\mathcal{H}}{\partial H} = 1, \qquad \frac{\partial\mathcal{H}}{\partial t} = 0,$$

$$\therefore \qquad \mathbb{X}_{\mathcal{H}} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i} + \frac{\partial}{\partial t}. \tag{2.2.2.5}$$

Thus, $\mathbb{X}_{\mathcal{H}}$ is the time-dependent Hamiltonian vector field. The vector field $\mathbb{X}_{\mathcal{H}}$ lies in the kernel of $d\theta_{\mathcal{H}}$, so the bicharacteristic of $\theta_{\mathcal{H}}$ is a path through the extended phase space such that the tangent vector to the path at any point is parallel to $\mathbb{X}_{\mathcal{H}}$.

It is clear that the Poincaré-Cartan two form associated to (2.2.2.2)

$$\omega = d\theta_{\mathcal{H}} = \sum_i dp_i \wedge dx^i - dH \wedge dt \tag{2.2.2.6}$$

is invariant under Jacobi-Maupertuis transformation. This reveals that the JM transformation is the *time-dependent canonical transformation.* Now consider the time-dependent canonical transformations of the extended phase space,

$$\begin{aligned} t &\to \sigma \qquad & d\sigma &= \Lambda(x,p)dt, \\ H &\to \widetilde{H} \qquad & \widetilde{H} &= \Lambda^{-1}(x,p)H. \end{aligned} \tag{2.2.2.7}$$

where $\Lambda(x,p) = (E - U(x))$. This changes the initial equations of motion

$$\frac{dx^i}{d\sigma} = \Lambda^{-1}(x,p)\left(\frac{dx^i}{dt} - \widetilde{H}\frac{\partial\Lambda}{\partial p_i}\right), \qquad \frac{dp_i}{d\sigma} = \Lambda^{-1}(x,p)\left(\frac{dp_i}{dt} + \widetilde{H}\frac{\partial\Lambda}{\partial x^i}\right).$$

This preserves the canonical form of the Hamilton-Jacobi equation given by

$$\frac{\partial S}{\partial\sigma} + \widetilde{H} = \frac{\partial S}{\partial t}\frac{dt}{d\sigma} + \Lambda^{-1}H = \Lambda^{-1}\left(\frac{\partial S}{\partial t} + H\right) = 0.$$

In other words, S satisfies

$$S = \int (p_i dx^i - H dt) = \int (p_i dx^i - \widetilde{H} d\sigma).$$

Integral trajectories have two parametric forms X_H and $X_{\widetilde{H}}$ corresponding to the Hamiltonians H and $\widetilde{H} = \Lambda^{-1}(x,p)H$ respectively. The transformation $X_H \to X_{\widetilde{H}}$ is the Maupertuis transformation. If σ be the time along trajectories of the vector $X_{\widetilde{H}}$, then the Maupertuis transformation gives the Jacobi transformation $d\sigma = (E - U(x))dt$.

Thus, the reparametrization can be seen as part of the canonical transformation [32, 33] to counter the changes in the form of the equation of motion. This maps the geodesic onto another geodesic while preserving integrability.

Naturally, for a transformed Hamiltonian, the dynamical description should also change to match the new generator of time translations. This essentially means that the geodesic must be reparameterized to keep the form of Hamilton's equations invariant. Furthermore, from the lifted Hamiltonian, using (2.2.1.5) and (2.2.1.7) gives the momentum and reparametrization factor:

$$\frac{dx^i}{d\sigma} = \frac{\partial \widetilde{H}}{\partial p_i} = 2\widetilde{g}^{ij}(x)p_j = \frac{1}{m\left(E - U(x)\right)}\left(g^{ij}(x)p_j\right) = \frac{dt}{d\sigma}\dot{x}^i,$$

$$p_i = \frac{1}{2}\widetilde{g}_{ij}(x)\frac{dx^j}{d\sigma}, \qquad \Lambda(x,p) = \frac{d\sigma}{dt} = \left|E - U(x)\right|. \tag{2.2.2.8}$$

Thus, according to (2.2.2.7), the new Hamiltonian can be said to be:

$$\mathcal{H} = \frac{H}{\left|E - U(x)\right|}. \tag{2.2.2.9}$$

Using (2.2.2.8) for the Jacobi Hamltonian, we can say that the reduced Lagrangian is

$$\widetilde{\mathcal{L}} = \widetilde{g}_{ij}(x)\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma} = 4\widetilde{g}^{ij}(x)\left(\frac{1}{2}\widetilde{g}_{ik}(x)\frac{dx^k}{d\sigma}\right)\left(\frac{1}{2}\widetilde{g}_{jl}(x)\frac{dx^l}{d\sigma}\right) = 4\widetilde{g}^{ij}(x)p_ip_j = 4\widetilde{H} = 4$$

$$\therefore \qquad \widetilde{\mathcal{L}} = \left(E - U(x)\right)\mathcal{L} \quad = \quad \widetilde{g}_{ij}(x)\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma} = 4 \tag{2.2.2.10}$$

Liouville integrability of an n-dimensional geodesic flow is defined to imply that:

a. n functionally independent first-integrals of motion I_n exist almost everywhere.

b. Such integrals are in involution: $\{I_j, I_k\} = 0$ for all $1 < j, k < n$.

Restricting the geodesic flow onto any non-zero fixed energy level surfaces are smoothly equivalent to the trajectory. Consequently, we may redefine the condition of integrability to imply the existence of $n-1$ functionally independent first integrals in involution almost everywhere on the unit covector bundle $\{\widetilde{H}(x,p) = \widetilde{g}^{ij}(x)p_ip_j = 1\} \subset T^*M^n$ [34].

2.2.3 Conserved quantities and Clairaut's constant

Starting with the Hamiltonian in (2.2.1.7), we shall write the dynamical equations with respect to a new parameter s as shown in [1, 16]

$$\begin{aligned} \frac{dx^i}{ds} &= \frac{\partial \widetilde{H}}{\partial p_i} = \frac{g^{ij}(\boldsymbol{x})}{2m\left(E - U(\boldsymbol{x})\right)}p_j, \\ \frac{dp_i}{ds} &= -\frac{\partial \widetilde{H}}{\partial x^i} = -\frac{1}{2m\left(E - U(\boldsymbol{x})\right)}\left[\frac{1}{2}\frac{\partial g^{ij}(\boldsymbol{x})}{\partial x^i}p_ip_j + \frac{\partial U}{\partial x^i}\right]. \end{aligned} \tag{2.2.3.1}$$

Upon comparison with (2.2.1.5), we can see that the dynamical equations are unaltered, except for a reparametrization as in [1, 16], given by:

$$\frac{ds}{dt} = 2m\left(E - U(\boldsymbol{x})\right). \tag{2.2.3.2}$$

Consequently, for any conserved quantity $K = K^{(2)ij}p_i p_j + K^{(0)}$, we can say:

$$\frac{dK}{ds} = \left\{K, \widetilde{H}\right\} = \frac{dt}{ds}\frac{dK}{dt} = \frac{1}{2m\left[E - U(\boldsymbol{x})\right]}\left\{K, H\right\}. \tag{2.2.3.3}$$

$$\therefore \qquad \left\{K, \widetilde{H}\right\} = 0 \qquad \Rightarrow \qquad \{K, H\} = 0. \tag{2.2.3.4}$$

In [31], T. Houri describes $\widetilde{K} = K^{(2)ij}p_i p_j + K^{(0)}\widetilde{H}$ where according to (2.2.1.7), we can say

$$\widetilde{K} = K^{(2)ij}p_i p_j + K^{(0)}\widetilde{H} \quad = \quad K^{(2)ij}p_i p_j + K^{(0)} = K, \qquad \because \quad \widetilde{H} = 1, \tag{2.2.3.5}$$

thus, showing that the conserved quantities remain the same for the Jacobi metric. This is not surprising given that the Jacobi-Eisenhart lift was just a reparametrization that left position and momenta unaltered. Since all conserved quantities or first integrals in Hamiltonian mechanics are polynomials of position and momenta, they should also be unchanged under such a transformation, unless a canonical transformation is involved.

Taking angular momentum for example, if the spatial metric exhibits spherical symmetry, as described below:

$$g_{ij}(\boldsymbol{x})dx^i dx^j = W^2(\boldsymbol{x})dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{2.2.3.6}$$

Then using (2.2.3.2), we will have the conserved angular momentum for $\theta = \frac{\pi}{2}$ in the form known as Clairaut's constant given by:

$$R = 2mr^2\Big(E - U(\boldsymbol{x})\Big)\frac{d\varphi}{ds} = mr^2\frac{d\varphi}{d\tau} = const. \tag{2.2.3.7}$$

showing that the angular momentum R in (2.2.3.7), as a first integral is invariant under such formulation as shown in (2.2.3.5).

2.3 Formulation from a metric line element

One of the authors formulated the Jacobi metric from the line element in [17] and demonstrated the formulation for the Schwarzschild metric. Here, we will show how the line element formulation equates to that given by (2.2.1.8) which describes the non-relativistic formulation.

It is worth noting that in [17], the Jacobi metric was formulated only for static metrics and stationary metrics of the Zermelo form. Here we formulate the Jacobi metric for stationary metrics of the Randers form of Finsler metric. Stationary metrics (with vector potential terms $A_i \neq 0$) are distinct from static metrics in the sense that while both are time-translation invariant, only static metrics are time-reversal invariant. If $A_i = 0$ stationary metrics reduce to static metrics.

Let us consider the following metric:

$$dl^2 = -c^2V^2(\boldsymbol{x})\left(dt + A_i(\boldsymbol{x})dx^i\right)^2 + g_{ij}(\boldsymbol{x})dx^i dx^j. \tag{2.3.1}$$

and the corresponding Lagrangian is given as:

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = m\sqrt{c^2V^2(\boldsymbol{x})\left(\dot{t} + A_i(\boldsymbol{x})\dot{x}^i\right)^2 - g_{ij}(\boldsymbol{x})\dot{x}^i\dot{x}^j}. \tag{2.3.2}$$

The momentum conjugate to co-ordinates are given by:

$$
\begin{aligned}
\frac{H}{c} &= \frac{\partial L}{\partial \dot{t}} = \frac{mc^2V^2(\boldsymbol{x})\left(\dot{t}+A_k(\boldsymbol{x})\dot{x}^k\right)}{\sqrt{c^2V^2(\boldsymbol{x})\left(\dot{t}+A_k(\boldsymbol{x})\dot{x}^k\right)^2-g_{ij}(\boldsymbol{x})\dot{x}^i\dot{x}^j}} = \frac{\mathcal{E}}{c},\\
\frac{p_i}{c} &= \frac{\partial L}{\partial \dot{x}^i} = \frac{m\left\{c^2V^2(\boldsymbol{x})A_i(\boldsymbol{x})\left(\dot{t}+A_k(\boldsymbol{x})\dot{x}^k\right)-g_{ij}(\boldsymbol{x})\dot{x}^j\right\}}{\sqrt{c^2V^2(\boldsymbol{x})\left(\dot{t}+A_k(\boldsymbol{x})\dot{x}^k\right)^2-g_{ij}(\boldsymbol{x})\dot{x}^i\dot{x}^j}}.
\end{aligned}
\tag{2.3.3}
$$

With the following calculations using (2.3.3), we will have

$$
\begin{aligned}
\left(\frac{\mathcal{E}}{c}\right)^2 - m^2c^2V^2(\boldsymbol{x}) &= m^2c^2V^2(\boldsymbol{x})\left[\frac{c^2V^2(\boldsymbol{x})\left(\dot{t}+A_k(\boldsymbol{x})\dot{x}^k\right)^2}{c^2V^2(\boldsymbol{x})\left(\dot{t}+A_k(\boldsymbol{x})\dot{x}^k\right)^2-g_{ij}(\boldsymbol{x})\dot{x}^i\dot{x}^j}-1\right],\\
&= \frac{m^2c^2V^2(\boldsymbol{x})g_{ij}(\boldsymbol{x})\dot{x}^i\dot{x}^j}{c^2V^2(\boldsymbol{x})\left(\dot{t}+A_k(\boldsymbol{x})\dot{x}^k\right)^2-g_{ij}(\boldsymbol{x})\dot{x}^i\dot{x}^j}.
\end{aligned}
$$

From (2.3.3), we can see that the gauge-covariant momenta are given by:

$$
\frac{\Pi_i}{c} = \frac{p_i}{c} - \frac{mc^2V^2(\boldsymbol{x})A_i(\boldsymbol{x})\left(\dot{t}+A_j\dot{x}^j\right)}{\sqrt{c^2V^2(\boldsymbol{x})\left(\dot{t}+A_k(\boldsymbol{x})\dot{x}^k\right)^2-g_{ij}(\boldsymbol{x})\dot{x}^i\dot{x}^j}} = \frac{-mg_{ij}(\boldsymbol{x})\dot{x}^j}{\sqrt{c^2V^2(\boldsymbol{x})\left(\dot{t}+A_k(\boldsymbol{x})\dot{x}^k\right)^2-g_{ij}(\boldsymbol{x})\dot{x}^i\dot{x}^j}},
$$

$$
\mathcal{E}^2 - m^2c^4V^2(\boldsymbol{x}) = c^2V^2(\boldsymbol{x})g^{ij}(\boldsymbol{x})\Pi_i\Pi_j \qquad \Rightarrow \qquad \frac{c^2V^2(\boldsymbol{x})g^{ij}(\boldsymbol{x})}{\mathcal{E}^2-m^2c^2V^2(\boldsymbol{x})}\Pi_i\Pi_j = 1. \tag{2.3.4}
$$

One can easily see that in the flat space setting $V^2(\boldsymbol{x}) = 1$ in (2.3.4), we have the familiar relativistic energy equation

$$
\mathcal{E}^2 = \left|\Pi\right|^2c^2 + m^2c^4.
$$

Thus from the inverse metric (2.3.4) we have the Jacobi metric given by:

$$
J^{ij}(\boldsymbol{x}) = \frac{c^2V^2(\boldsymbol{x})g^{ij}(\boldsymbol{x})}{\mathcal{E}^2-m^2c^4V^2(\boldsymbol{x})} \qquad \Rightarrow \qquad J_{ij}(\boldsymbol{x}) = \frac{\mathcal{E}^2-m^2c^4V^2(\boldsymbol{x})}{c^2V^2(\boldsymbol{x})}g_{ij}(\boldsymbol{x}). \tag{2.3.5}
$$

Thus, for a fixed relativistic energy $\mathcal{E}$, all timelike geodesics are geodesics of the above Jacobi metric. Now that we have summarized the formulation of the Jacobi metric for time-like geodesics, we shall see how it evolves under the non-relativistic approximation. Suppose that we write the temporal metric component as

$$
V^2(\boldsymbol{x}) = 1 + \frac{2U(\boldsymbol{x})}{mc^2}. \tag{2.3.6}
$$

and set the non-relativistic approximation rules

$$
2U(\boldsymbol{x}) << mc^2 \qquad\qquad g^{ij}(\boldsymbol{x})\Pi_i\Pi_j << m^2c^2. \tag{2.3.7}
$$

From 2.3.4, we can see that on applying (2.3.6) and (2.3.7), we get

$$
\begin{aligned}
\mathcal{E} &= mc^2\sqrt{1+\frac{2U(\boldsymbol{x})}{mc^2}}\sqrt{1+\frac{g^{ij}(\boldsymbol{x})\Pi_i\Pi_j}{m^2c^2}}\\
&\approx \left(1+\frac{U(\boldsymbol{x})}{mc^2}+...\right)\left(mc^2+\frac{1}{2}\frac{g^{ij}(\boldsymbol{x})\Pi_i\Pi_j}{m}+...\right) = mc^2+\frac{1}{2}\frac{g^{ij}(\boldsymbol{x})\Pi_i\Pi_j}{m}+U(\boldsymbol{x})+....,
\end{aligned}
$$

$$\therefore \qquad \mathcal{E} \approx mc^2 + \frac{1}{2}\frac{g^{ij}(\boldsymbol{x})\Pi_i\Pi_j}{m} + U(\boldsymbol{x}) = mc^2 + T + U(\boldsymbol{x}).$$

which assures us that our approximation is on the right track. We shall now rewrite the energy in the following form:

$$\begin{aligned} &\mathcal{E} \approx mc^2 + E \qquad E = T + U(\boldsymbol{x}) << mc^2, \\ &\left(\frac{\mathcal{E}}{mc^2}\right)^2 = \left(1 + \frac{E}{mc^2}\right)^2 \approx 1 + \frac{2E}{mc^2}. \end{aligned} \tag{2.3.8}$$

We will now see that the Jacobi metric as demonstrated in (2.3.5) under the approximations of (2.3.7) and (2.3.8) becomes

$$\begin{aligned} J_{ij}(\boldsymbol{x}) &= \frac{\mathcal{E}^2 - m^2c^4V^2(\boldsymbol{x})}{c^2V^2(\boldsymbol{x})} g_{ij}(\boldsymbol{x}) = \frac{\left(\frac{\mathcal{E}}{mc^2}\right)^2 - V^2(\boldsymbol{x})}{\left(\frac{V(\boldsymbol{x})}{mc}\right)^2} g_{ij}(\boldsymbol{x}) \approx \frac{\left(1 + \frac{2E}{mc^2}\right) - \left(1 + \frac{2U(\boldsymbol{x})}{mc^2}\right)}{\frac{1}{(mc)^2}\left(1 + \frac{2U(\boldsymbol{x})}{mc^2}\right)} g_{ij}(\boldsymbol{x}) \\ &= \frac{2m\left(E - U(\boldsymbol{x})\right)}{\left(1 + \frac{2U(\boldsymbol{x})}{mc^2}\right)} g_{ij}(\boldsymbol{x}) \approx 2m\left(E - U(\boldsymbol{x})\right) g_{ij}(\boldsymbol{x}), \end{aligned}$$

$$\therefore \qquad \boxed{J_{ij}(\boldsymbol{x}) = 2m\left(E - U(\boldsymbol{x})\right) g_{ij}(\boldsymbol{x})}. \tag{2.3.9}$$

Thus, the Jacobi metric in the non-relativistic approximations agrees with the result (2.2.1.8), showing that both formulations of a projection of the geodesic onto the constant energy hypersurface are consistent and correct.

2.4 Jacobi metric for time-like geodesics in stationary space-time

Now that we have summarized the formulation of the Jacobi metric for time-like geodesics, we shall first demonstrate Gibbons' application for the formulation on the Schwarzschild metric [17], Then we shall proceed to apply the present formulation of the Jacobi-metric by the to other static and *stationary* space-time metrics such as Taub-NUT, Bertrand and Kerr metrics.

2.4.1 Schwarzschild metric

For the Schwarzschild metric (setting $c = 1$) we are dealing with the case where $A_i(\boldsymbol{x}) = 0$ given by:

$$dl^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\ d\varphi^2\right). \tag{2.4.1.1}$$

We can say that

$$V^2(\boldsymbol{x}) = \left(1 - \frac{2M}{r}\right) \qquad g_{ij}dx^idx^j = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\ d\varphi^2\right). \tag{2.4.1.2}$$

Thus, the relativstic Schwarzschild Jacobi metric according to (2.3.5) is given by

$$J_{ij}(\boldsymbol{x})dx^i dx^j = \left[\mathcal{E}^2 - m^2\left(1-\frac{2M}{r}\right)\right]\left[\left(1-\frac{2M}{r}\right)^{-2}dr^2 + \left(1-\frac{2M}{r}\right)^{-1}r^2\left(d\theta^2+\sin^2\theta\ d\varphi^2\right)\right]. \tag{2.4.1.3}$$

and the non-relativistic Schwarzschild Jacobi metric according to (2.2.1.8) is given by

$$\widetilde{g}_{ij}(\boldsymbol{x})dx^i dx^j = 2m\left[E+\frac{mM}{r}\right]\left[\left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2+\sin^2\theta\ d\varphi^2\right)\right], \tag{2.4.1.4}$$

Thus, concuding Gibbons' example of application of the formulation to a static metric [17]. Now we shall formulate the Jacobi metric for other geodesics in *stationary space-times*.

2.4.2 The Taub-NUT metric

In 1951, Abraham Huskel Taub found an exact solution of Einstein's equations, which was subsequently extended to a larger manifold by E. Newman, T. Unti and L. Tamburino in 1963, known as the the Taub-NUT [35]. It is a gravitational anti-instanton with corresponding SU(2) gauge fields, with geodesics which approximately describe the motion of well separated monopole-monopole interactions. As a dynamical system it exhibits spherically symmetry, with geodesics admitting Kepler-type symmetry.

The Euclidean Taub-NUT metric is given by:

$$dl^2 = 4M^2\frac{r-M}{r+M}\left(d\psi+\cos\theta\ d\varphi\right)^2 + \frac{r+M}{r-M}dr^2 + \left(r^2-M^2\right)\left(d\theta^2+\sin^2\theta\ d\varphi^2\right). \tag{2.4.2.1}$$

where $\psi \equiv t$. However, it is not a space-time due to the Euclidean signature, which results in a slightly different form of Jacobi metric derived by the same approach. Furthermore, the nature of its potential term distinguishes it from other space-times, such that the lower energy and weak potential limits (for other space-times we shall see that $V^2(\boldsymbol{x})_{M=0} = 1$) need to be differently defined. Here, we can see that

$$\begin{aligned} V^2(\boldsymbol{x}) &= 4M^2\frac{r-M}{r+M} \qquad\qquad V^2(\boldsymbol{x})_{M=0} = 0, \\ g_{ij}dx^i dx^j &= \frac{r+M}{r-M}dr^2 + \left(r^2-M^2\right)\left(d\theta^2+\sin^2\theta\ d\varphi^2\right). \end{aligned} \tag{2.4.2.2}$$

Thus, the geometric line-element based Jacobi metric derived in the same manner as (2.3.5) is given by

$$J_{ij}(\boldsymbol{x})dx^i dx^j = \frac{(r+M)^2}{4M^2}\left(4m^2M^2\frac{r-M}{r+M} - \mathcal{Q}^2\right)\left[\frac{dr^2}{(r-M)^2} + \left(d\theta^2+\sin^2\theta\ d\varphi^2\right)\right]. \tag{2.4.2.3}$$

where $\mathcal{Q} = m\frac{\partial}{\partial\psi}\sqrt{\left(\frac{dl}{d\tau}\right)^2}$. On the other hand, the Lagrangian based Jacobi metric derived in the same manner as (2.2.1.8) (according to (2.2.3), $E = \sum_\mu p_\mu \dot{x}^\mu - L_{geod} = 0$) is given by

$$\widetilde{g}_{ij}(\boldsymbol{x})dx^i dx^j = -Q^2\frac{(r+M)^2}{4M^2}\left[\frac{dr^2}{(r-M)^2} + \left(d\theta^2+\sin^2\theta\ d\varphi^2\right)\right]. \tag{2.4.2.4}$$

which describes the weak potential limit $V^2(\boldsymbol{x}) \approx 0$, where $Q = \frac{m}{2}\frac{\partial}{\partial \dot{\psi}}\left[\left(\frac{dl}{d\tau}\right)^2\right]$ is a conserved quantity. Now we shall turn our attention to another case: the Bertrand space-time metric.

2.4.3 The Bertrand space-time metric

According to Bertrand's theorem, all bounded, closed and periodic orbits in Euclidean space are associated only with two potentials: the Kepler-Coloumb $U(r) = \frac{a}{r} + b$ and the Hooke-Oscillator $U(r) = ar^2 + b$, which are dual to each other, related via the Bohlin-Arnold-Vasiliev transformation [36, 37]. The Taub-NUT metric previously discussed is effectively a Euclidean Bertrand space-time metric with magnetic fields applied and exhibits the same duality as shown in [37]. Perlick showed that Bertrand's theorem arises in General Relativity as well [38]. The Bertrand space-time metric is given as:

$$dl^2 = -\frac{dt^2}{\Gamma(r)} + h^2(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta\ d\varphi^2\right). \tag{2.4.3.1}$$

Since angular momentum is conserved under spherical symmetry, taking $\theta = \frac{\pi}{2}$ and defining $\frac{1}{\Gamma(r)} = 1 + \frac{2U(r)}{m}$, the natural Hamiltonian is given as:

$$H(\boldsymbol{x}, \boldsymbol{p}) = \frac{p_r^2}{2h^2(r)} + \frac{p_\varphi^2}{2r^2} + \frac{m}{2}\left(\frac{1}{\Gamma(r)} - 1\right) = E. \tag{2.4.3.2}$$

Therefore, the Hamilton's dynamical equations are:

$$\begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{h^2(r)} \qquad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_r^2}{h^2(r)}\frac{h'(r)}{h(r)} + \frac{p_\varphi^2}{r^3} + \frac{m\Gamma'(r)}{2\Gamma^2(r)}, \\ \therefore \qquad \dot{p}_r &= \left(2E + m - \frac{m}{\Gamma(r)}\right)\frac{h'(r)}{h(r)} + \left(\frac{1}{r} - \frac{h'(r)}{h(r)}\right)\frac{p_\varphi^2}{r^2} + \frac{m\Gamma'(r)}{2\Gamma^2(r)}. \end{aligned} \tag{2.4.3.3}$$

The radial equation of motion is:

$$\ddot{r} = -\left(2E + m - \frac{1}{\Gamma(r)}\right)\frac{h'(r)}{h^3(r)} + \left(\frac{1}{r} + \frac{h'(r)}{h(r)}\right)\frac{p_\varphi^2}{h^2(r)r^2} + \frac{m\Gamma'(r)}{2h^2(r)\Gamma^2(r)}.$$

which for the Kepler problem $U(r) = -\frac{k}{r}, h^2(r) = 1$ is:

$$\ddot{r} = \frac{p_\varphi^2}{r^3} - \frac{k}{r^2}.$$

By regular formulation, the Jacobi metric is given as:

$$\widetilde{g}_{ij}(\boldsymbol{x})dx^i dx^j = \left[E + \frac{m}{2}\left(1 - \frac{1}{\Gamma(r)}\right)\right]\left[h^2(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta\ d\varphi^2\right)\right]. \tag{2.4.3.4}$$

for which the reparameterized Hamilton's equations according to (2.2.3.1) are:

$$\frac{dr}{ds} = \frac{dt}{ds}\dot{r} = \frac{2\Gamma(r)}{(2E+m)\Gamma(r) - m}\frac{p_r}{h^2(r)}. \tag{2.4.3.5}$$

$$\frac{dp_r}{ds} = \frac{dt}{ds}\dot{p}_r = \frac{2\Gamma(r)}{(2E+m)\Gamma(r) - m}\left[\left(2E + m - \frac{m}{\Gamma(r)}\right)\frac{h'(r)}{h(r)} + \left(\frac{1}{r} - \frac{h'(r)}{h(r)}\right)\frac{p_\varphi^2}{r^2} + \frac{m\Gamma'(r)}{2\Gamma^2(r)}\right]. \tag{2.4.3.6}$$

For example, if we consider the Kepler problem, we set $U(r) = -\frac{k}{r}, h^2(r) = 1$ and we have:

$$\frac{dr}{ds} = \frac{dt}{ds}\dot{r} = \frac{2r}{2Er + k}p_r. \tag{2.4.3.7}$$

$$\frac{dp_r}{ds} = \frac{dt}{ds}\dot{p}_r = \frac{2r}{2Er + k}\left(\frac{p_\varphi^2}{r^3} - \frac{k}{r^2}\right). \tag{2.4.3.8}$$

However, if we were to apply the treatment for time-like geodesics of [17] where $c^2V^2(r) = \frac{1}{\Gamma(r)}$, then we would have the time-like Jacobi Bertrand metric as per (2.3.5) is

$$J_{ij}(\boldsymbol{x})dx^i dx^j = \left(\mathcal{E}^2\Gamma(r) - m^2c^2\right)\left[h^2(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta\ d\varphi^2\right)\right]. \tag{2.4.3.9}$$

The next metric we shall deal with is the Kerr metric.

2.4.4 The Kerr metric

In [39], the Jacobi metric of the Reissner-Nördstrom space-time was given. Here, we shall turn our attention to another black-hole space-time known as the rotating (Kerr) black hole. This is a stationary metric.

The Kerr metric (setting $c = 1$) is:

$$\begin{aligned} dl^2 = &-\left(1 - \frac{2GMr}{\rho^2}\right)dt^2 - \frac{4GMar\sin^2\theta}{\rho^2}d\phi\ dt \\ &+ \frac{\rho^2}{\Delta}dr^2 + \rho^2\ d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right)^2 - a^2\Delta\sin^2\theta\right]d\phi^2, \\ &\Delta(r) = r^2 - 2GMr + a^2 \qquad \rho^2(r,\theta) = r^2 + a^2\cos^2\theta. \end{aligned} \tag{2.4.4.1}$$

Here, the potential term $V^2(\boldsymbol{x})$ and the the spatial metric $g_{ij}(\boldsymbol{x})$ are taken to be

$$V^2(\boldsymbol{x}) = 1 - \frac{2GMr}{\rho^2}. \tag{2.4.4.2}$$

$$g_{ij}(\boldsymbol{x}) = \frac{\rho^2}{\Delta}dr^2 + \rho^2\ d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right)^2 - a^2\Delta\sin^2\theta\right]d\phi^2. \tag{2.4.4.3}$$

So, using the potential (2.4.4.2) and the relativistic Jacobi metric formulation 2.3.5 gives us

$$J_{ij}(\boldsymbol{x})dx^i dx^j = \left(\frac{\mathcal{E}^2\rho^2}{\rho^2 - 2GMr} - m^2\right)\left[\frac{\rho^2}{\Delta}dr^2 + \rho^2\ d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left\{\left(r^2 + a^2\right)^2 - a^2\Delta\sin^2\theta\right\}d\phi^2\right]. \tag{2.4.4.4}$$

while the non-relativistic Jacobi metric formulation 2.2.1.8 gives us

$$\widetilde{g}_{ij}(\boldsymbol{x})dx^i dx^j = \left(E + \frac{2GMr}{\rho^2}\right)\left[\frac{\rho^2}{\Delta}dr^2 + \rho^2\ d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left\{\left(r^2 + a^2\right)^2 - a^2\Delta\sin^2\theta\right\}d\phi^2\right]. \tag{2.4.4.5}$$

Now we shall consider how to execute such a formulation for time-dependent systems.

2.5 Jacobi metric for time-dependent systems

Time dependent systems are essentially those where we find that the system energy is not conserved. Usually such systems are dissipative in nature. When we formulate the Jacobi metric for autonomous or time-independent systems, we are essentially projecting the geodesic to a constant energy hypersurface. However, a non-autonomous or time-dependent system does not possess a fixed energy hypersurface, requiring us to improvise our approach. One way to deal with time-dependent systems is the Eisenhart-Duval lift.

The Eisenhart-Duval lift, developed by L.P. Eisenhart [28] and rediscovered by C. Duval [29], with applications demonstrated in [40, 41] embeds non-relativistic theories into Lorentzian geometry. It is one example of a method for geometrizing interactions, where a classical system in n dimensions is shown to be dynamically equal to a Lorentzian $n+2$ space-time. It provides a relativistic framework to study nonrelativistic physics, simplifying the study of symmetries of a Hamiltonian system by looking at geodesic Hamiltonians. The hidden symmetries of this lift were studied from the perspective of the Dirac equation by Cariglia [42], and it was applied to study the projective and conformal symmetries and quantisation of dissipative systems such as Caldirola and Kannais damped simple harmonic oscillator in [43].

Let (M, g) be a pseudo-Riemannian manifold, ie. g is a non-degenerate symmetric two times covariant tensor field on M. Given a local chart $(U, x^1, ...x^n)$ on M, the local expression for g is is given by:

$$g = g_{ij}(\boldsymbol{x})dx^i \otimes dx^j.$$

and the corresponding metric is

$$dl^2 = g_{ij}(\boldsymbol{x})dx^i dx^j. \tag{2.5.1}$$

The geodesic of the equation is

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0.$$

where the connection

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}(\boldsymbol{x})\left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right).$$

can be obtained from the Euler-Lagrange equation from the Lagrangian L for a free particle, ie.

$$L = T_g = \frac{1}{2}g_{ij}(\boldsymbol{x})\dot{x}^i\dot{x}^j. \tag{2.5.2}$$

We define Lagrangians of the mechanical type for systems with configuration space M, $L \in C^\infty(TM)$, by choosing a pseudo-Riemannian structure g on M and a potential function $V \in C^\infty(TM)$ as follows

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \frac{1}{2}g_x(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) - V(\boldsymbol{x}) = \frac{1}{2}g_{ij}(\boldsymbol{x})\dot{x}^i\dot{x}^j - V(\boldsymbol{x}).$$

The key concept of the Eisenhart lift is to introduce a new degree of freedom with a new co-ordinate, thus replacing configuration space M with $\mathbb{R} \times M$. Eisenhart demonstrated the possibility of relating the dynamical trajectories of a Lagrangian mechanical system with a projection on M of extremal length curves on an extended manifold $\widetilde{M} = \mathbb{R} \times M$ with the Riemannian structure

$$\widetilde{g} = \Pi_2^* g - \frac{1}{2V}dz \otimes dz.$$

where

$$\Pi_{1,2} : \mathbb{R} \times M \longrightarrow \mathbb{R}, M.$$

If we assume g_{00} as a function A of the co-ordinates $(x^1, ...x^n)$, the square of arc length geometry

$$ds^2 = g_{ij}(\boldsymbol{x})dx^i dx^j + A(\boldsymbol{x})dz^2.$$

with the associated motion geometry

$$T_g = \frac{1}{2}\Big(g_{ij}(\boldsymbol{x})\dot{x}^i\dot{x}^j + A(\boldsymbol{x})\dot{z}^2\Big). \tag{2.5.3}$$

then the equations of motion in terms of arc-length s is given by

$$x^{i\prime\prime} + \Gamma^i_{jk}x^{j\prime}x^{k\prime} - g^{ij}\frac{\partial A}{\partial x^j}(z')^2 = 0.$$

Since z is a cyclical variable, we should have

$$A(\boldsymbol{x})\dot{z} = c \in \mathbb{R}.$$

For each value of the parameter c, we can use a new parameter $t = cs$. Then the differential equations reduce to

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k - g^{ij}\frac{1}{2A^2}\frac{\partial A}{\partial x^j} = 0 \qquad\qquad A(\boldsymbol{x})\dot{z} = 1.$$

Note that when $c = 1$, the parameter t coincides with s, and the condition $A(\boldsymbol{x})\dot{z} = 1$ corresponds to $p_z = 1$. If we choose $A = (2V)^{-1}$, then we obtain

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k + g^{ij}\frac{\partial V}{\partial x^j} = 0. \tag{2.5.4}$$

Thus, $\widetilde{g}$ is associated with kinetic energy (2.5.3) after Legendre transform leads to the new Hamiltonian.

$$H = \frac{1}{2}\left(g^{ij}p_ip_j + Vp_z^2\right). \tag{2.5.5}$$

which coincides with the natural Hamiltonian of mechanical type for $p_z = \sqrt{2}$. One way to understand how it makes a difference is shown in the following subsections.

2.5.1 The Metric without Eisenhart Lift

We shall first look at the look at the system portrayed originally without the Eisenhart lift. If the given general metric without Eisenhart lift is:

$$dl^2 = h_{ij}(\boldsymbol{x}, t)dx^i dx^j + 2\frac{A_i(\boldsymbol{x}, t)}{m}cdx^i dt - 2\frac{\Phi(\boldsymbol{x}, t)}{m}\ c^2dt^2.$$

then the Lagrangian is given by

$$L = \frac{m}{2}h_{ij}(\boldsymbol{x}, t)\dot{x}^i\dot{x}^j + A_i(\boldsymbol{x}, t)c\dot{x}^i\dot{t} - \Phi(\boldsymbol{x}, t)\ c^2\dot{t}^2.$$

We will have the momenta

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = mh_{ij}(\boldsymbol{x}, t)\dot{x}^j + A_i(\boldsymbol{x}, t)c\dot{t} \qquad\qquad p_t = \frac{\partial L}{\partial \dot{t}} = A_i(\boldsymbol{x}, t)c\dot{x}^i - 2\Phi(\boldsymbol{x}, t)c^2\dot{t} = -H.$$

The Maupertuis form of the action gives the Lagrangian along the geodesic (2.2.2.4). Thus, we will have at least one conserved quantity which is the overall Legendre Hamiltonian:

$$\frac{dL_{geod}}{d\tau} = \frac{\partial L_{geod}}{\partial x^\mu}\dot{x}^\mu + \frac{\partial L_{geod}}{\partial \dot{x}^\mu}\ddot{x}^\mu = \underbrace{\left[\frac{\partial L_{geod}}{\partial x^\mu} - \frac{d}{d\tau}\left(\frac{\partial L_{geod}}{\partial \dot{x}^\mu}\right)\right]}_{0}\dot{x}^\mu + \frac{d}{d\tau}\left(\frac{\partial L_{geod}}{\partial \dot{x}^\mu}\dot{x}^\mu\right),$$

$$\Rightarrow \quad \frac{d}{d\tau}\left(\frac{\partial L_{geod}}{\partial \dot{x}^\mu}\dot{x}^\mu - L_{geod}\right) = 0 \qquad \Rightarrow \qquad \mathcal{H} = \frac{\partial L_{geod}}{\partial \dot{x}^\mu}\dot{x}^\mu - L_{geod} = 0 = conserved.$$

Now, depending on the metric's dependence on time, we will face different situations.

Time-Independent Case

When independent of time t, we will have another conserved quantity H in addition to $\mathcal{H}$

$$-H = \frac{\partial L}{\partial \dot{t}} = conserved.$$

From (2.2.2.4), we can see that this conserved quantity under time parametrization ($\dot{t} = 1$) is given by

$$H = p_i\dot{x}^i - L_{geod} = \frac{1}{2m}h^{ij}(\boldsymbol{x})(p_i - cA_i)(p_j - cA_j) + \Phi(\boldsymbol{x}).$$

Thus, we have 2 conserved quantities: $\mathcal{H}$ and H.

Time-Dependent Case

If the metric is time-dependent, H will not be a conserved quantity. This means that we are forced to resort to $\mathcal{H}$ as the only conserved quantity.

2.5.2 The Metric with Eisenhart Lift

This time, we will modify the metric with the Eisenhart lift by introducing a dummy variable σ. If the given general metric with Eisenhart lift is:

$$dl^2 = h_{ij}(\boldsymbol{x}, t)dx^i dx^j + 2c\ dt\ d\sigma + 2\frac{A_i(\boldsymbol{x}, t)}{m}dx^i dt - \frac{2\Phi(\boldsymbol{x}, t)}{m}\ c^2 dt^2.$$

where the metric is independent of σ, then the Lagrangian is given by

$$L = \frac{m}{2}h_{ij}(\boldsymbol{x}, t)\dot{x}^i\dot{x}^j + mc\dot{t}\dot{\sigma} + A_i(\boldsymbol{x}, t)c\dot{x}^i\dot{t} - \Phi(\boldsymbol{x}, t)\ c^2\dot{t}^2. \tag{2.5.2.1}$$

We will have the momenta, where one is a conserved quantity

$$p_i = \frac{\partial L}{\partial \dot{x}^i} \qquad p_t = \frac{\partial L}{\partial \dot{t}} = mc\dot{\sigma} + A_i(\boldsymbol{x}, t)c\dot{x}^i - 2\Phi(\boldsymbol{x}, t)c^2\dot{t} \qquad p_\sigma = \frac{\partial L}{\partial \dot{\sigma}} = mc\dot{t} = conserved.$$

The Maupertuis form of the action gives the Lagrangian along the geodesic as:

$$L_{geod} = p_\mu\dot{x}^\mu = \frac{\partial L_{geod}}{\partial \dot{x}^\mu}\dot{x}^\mu = p_i\dot{x}^i + p_t\dot{t} + p_\sigma\dot{\sigma} = p_i\dot{x}^i + p_\sigma\dot{\sigma} + \frac{p_t p_\sigma}{mc}. \tag{2.5.2.2}$$

As before, we will have the overall Legendre Hamiltonian $\mathcal{H}$ as a conserved quantity. Now we look at the cases of the metric's dependence on time.

Time-Independent Case

When independent of time t, as before we have another conserved quantity p_t. From (2.5.2.2), we can see that this conserved quantity is given by

$$-\frac{p_\sigma p_t}{mc} = p_i\dot{x}^i + p_\sigma\dot{\sigma} - L_{geod} = \frac{1}{2m}h^{ij}(\boldsymbol{x})\left(p_i - \frac{p_\sigma}{m}A_i\right)\left(p_j - \frac{p_\sigma}{m}A_j\right) + \Phi(\boldsymbol{x})\left(\frac{p_\sigma}{m}\right)^2 = H.$$

Thus, we have 3 conserved quantities: $\mathcal{H}$, H and p_σ.

Time-Dependent Case

If the metric is time-dependent, H will not be a conserved quantity. This means that

$$\mathcal{H} = \left(p_i\dot{x}^i + p_\sigma\dot{\sigma} - L_{geod}\right) + p_t\dot{t} = H + p_t\dot{t} = H + \frac{p_t p_\sigma}{mc} = 0,$$

$$\therefore \quad p_\sigma = -\frac{mcH}{p_t} = conserved, \tag{2.5.2.3}$$

showing that we have 2 conserved quantities: $\mathcal{H}$ and p_σ.

Thus, we can say that the Eisenhart-Duval lift is a useful tool for dealing with time-dependent systems by giving another conserved quantity p_σ to replace the natural Hamiltonian H normally used to parameterize motion on the cotangent space.

2.5.3 Formulation

In this section, we will demonstrate the deduction of the Jacobi-metric for time-dependent systems. The formulation has been deduced only with the metric line element.

Consider the following space-time metric:

$$dl^2 = c^2V^2(\boldsymbol{x},t)dt^2 + 2c\ d\sigma\ dt - g_{ij}(\boldsymbol{x},t)dx^i dx^j. \tag{2.5.3.1}$$

Its corresponding line-element Lagrangian is given as:

$$L(\boldsymbol{x},\dot{\boldsymbol{x}},t) = m\sqrt{c^2V^2(\boldsymbol{x},t)\dot{t}^2 + 2c\dot{\sigma}\dot{t} - g_{ij}(\boldsymbol{x},t)\dot{x}^i\dot{x}^j}. \tag{2.5.3.2}$$

and the momentum conjugate to co-ordinates are given by:

$$\begin{aligned}
\frac{p_t}{c} &= \frac{\partial L}{\partial \dot{t}} = \frac{m\left[c^2V^2(\boldsymbol{x},t)\dot{t} + c\dot{\sigma}\right]}{\sqrt{c^2V^2(\boldsymbol{x},t)\dot{t}^2 + 2c\dot{\sigma}\dot{t} - g_{ij}(\boldsymbol{x},t)\dot{x}^i\dot{x}^j}}, \\
\frac{p_i}{c} &= \frac{\partial L}{\partial \dot{x}^i} = \frac{-mg_{ij}(\boldsymbol{x},t)\dot{x}^j}{\sqrt{c^2V^2(\boldsymbol{x},t)\dot{t}^2 + 2c\dot{\sigma}\dot{t} - g_{ij}(\boldsymbol{x},t)\dot{x}^i\dot{x}^j}}, \\
\frac{p_\sigma}{c} &= \frac{\partial L}{\partial \dot{\sigma}} = \frac{mc\dot{t}}{\sqrt{c^2V^2(\boldsymbol{x},t)\dot{t}^2 + 2c\dot{\sigma}\dot{t} - g_{ij}(\boldsymbol{x},t)\dot{x}^i\dot{x}^j}} = q.
\end{aligned} \tag{2.5.3.3}$$

Using the equation for the Maupertuis form of the action

$$L_{geod} = p_i\dot{x}^i + p_t\dot{t} + p_\sigma\dot{\sigma}.$$

we can deduce that for the line element, the relativistic energy equation is:

$$2cp_t p_\sigma = c^2 g^{ij} p_i p_j + c^2 V^2(\boldsymbol{x}, t) p_\sigma^2 + m^2 c^4 = Q^2. \tag{2.5.3.4}$$

With the following calculations we find that

$$\begin{aligned}
g^{ij}(\boldsymbol{x}, t) p_i p_j &= \frac{m^2 c^2 g_{ij}(\boldsymbol{x}, t) \dot{x}^i \dot{x}^j}{c^2 V^2(\boldsymbol{x}, t) \dot{t}^2 + 2c\dot{\sigma}\dot{t} - g_{ij}(\boldsymbol{x}, t) \dot{x}^i \dot{x}^j}, \\
\Rightarrow \qquad m^2 c^2 + g^{ij}(\boldsymbol{x}, t) p_i p_j &= \frac{m^2 c^2 \left(c^2 V^2(\boldsymbol{x}, t) \dot{t}^2 + 2c\dot{\sigma}\dot{t}\right)}{c^2 V^2(\boldsymbol{x}, t) \dot{t}^2 + 2c\dot{\sigma}\dot{t} - g_{ij}(\boldsymbol{x}, t) \dot{x}^i \dot{x}^j} = 2qp_t - q^2 c^2 V^2(\boldsymbol{x}, t), \\
\Rightarrow \qquad \frac{g^{ij}(\boldsymbol{x}, t)}{2qp_t - q^2 c^2 V^2(\boldsymbol{x}, t) - m^2 c^2} p_i p_j &= 1.
\end{aligned}$$

this result can be written by writing $V^2(\boldsymbol{x}, t) = 2mU(\boldsymbol{x}, t)$ as:

$$\frac{c^2 g^{ij}(\boldsymbol{x}, t)}{2\left[c^2 q p_t - q^2 c^4 U(\boldsymbol{x}, t)\right] - m^2 c^4} p_i p_j = 1.$$

Thus the time-dependent Jacobi-metric is given by:

$$\begin{aligned}
J^{ij}(\boldsymbol{x}, t) &= \frac{g^{ij}(\boldsymbol{x}, t)}{2\left[qp_t - q^2 U(\boldsymbol{x}, t)\right] - m^2 c^2}, \\
J_{ij}(\boldsymbol{x}, t) &= \left[2\left\{qp_t - q^2 U(\boldsymbol{x}, t)\right\} - m^2 c^2\right] g_{ij}(\boldsymbol{x}, t).
\end{aligned} \tag{2.5.3.5}$$

which projects the geodesic onto the constant momentum hypersurface $\frac{p_\sigma}{c} = q$. If we employ the following approximation:

$$Q = mc^2 + \mathcal{E}(t) \quad \Rightarrow \quad \left(\frac{Q}{mc^2}\right)^2 \approx 1 + \frac{2\mathcal{E}(t)}{mc^2} \qquad (\mathcal{E} << mc^2),$$

$$\therefore \qquad 2qp_t \approx m^2 c^2 + 2m\mathcal{E}(t). \tag{2.5.3.6}$$

then the Jacobi metric under (2.5.3.6) will approximate to:

$$J_{ij}(\boldsymbol{x}, t) = 2m\left[\mathcal{E}(t) - q^2 U(\boldsymbol{x}, t)\right] g_{ij}(\boldsymbol{x}, t). \tag{2.5.3.7}$$

which is the non-relativistic approximation for time-dependent systems modified by an Eisenhart-Duval lift. One may attempt to verify this by deducing the formulation starting from the mechanical Lagrangian (2.5.2.1), or use a projective transformation as shown in the next section.

2.6 Comparison to Projective Transformation

Projective geometry can be used to describe natural Hamiltonian systems and generate the dualities between them. The Jacobi metric can be alternatively formulated from a projective transformation in the phase space as described in [44]. This is described by the null Hamiltonian, for which the curve is parameterized by the arc length.

$$\mathcal{H} = \frac{1}{2m} g^{ij}(\boldsymbol{x}) p_i p_j + U(\boldsymbol{x}) p_u^2 - \mathrm{sgn}(H) p_y^2. \tag{2.6.1}$$

Upon setting $p_u^2 = 1$ and $p_y^2 = |E|$, where E is the energy, we get null geodesics that project down to the original system. Rescaling the Hamiltonian by the factor $\Omega^2 = E - V(\boldsymbol{x})$, gives

$$\widetilde{\mathcal{H}} = \frac{\mathcal{H}}{\Omega^2} = \frac{1}{2m}\frac{g^{ij}(\boldsymbol{x})p_i p_j}{E - U(\boldsymbol{x})} - 1.$$

This is the null geodesic Hamiltonian related to the Jacobi metric for time-independent systems. From it, the inverse metric, we can deduce the Jacobi metric (2.2.1.8).

$$\widetilde{g}^{ij}(\boldsymbol{x}) = \frac{1}{2m}\frac{g^{ij}(\boldsymbol{x})}{E - U(\boldsymbol{x})} \qquad \widetilde{g}_{ij}(\boldsymbol{x}) = 2m\,[E - U(\boldsymbol{x})]\,g_{ij}(\boldsymbol{x}). \tag{2.6.2}$$

To account for time-dependence, as previously we modify the null Hamiltonian (2.6.1) via an Eisenhart-Duval such that $-\mathrm{sgn}(H)p_y^2 \longrightarrow \frac{p_u p_v}{mc}$ to include the extra co-ordinate as a dummy variable

$$\mathcal{H} = \frac{1}{2m}g^{ij}(\boldsymbol{x},t)p_i p_j + U(\boldsymbol{x},t)p_u^2 + \frac{p_u p_v}{mc}. \tag{2.6.3}$$

which is essentially an Eisenhart-Duval lifted Hamiltonian. As before, if we rescale (2.6.3) by a factor $\Omega^2 = -p_u p_v - U(\boldsymbol{x},t)p_u^2$, then we will get the corresponding null geodesic Hamiltonian for the Jacobi metric.

$$\widetilde{\mathcal{H}} = \frac{\mathcal{H}}{\Omega^2} = -\frac{1}{2m}\frac{g^{ij}(\boldsymbol{x},t)p_i p_j}{p_u p_v + U(\boldsymbol{x},t)p_u^2} - 1.$$

Here, if we write $p_u = q = mc$, and $p_v = -\mathcal{E}(t)$ in accordance with (2.5.2.3), we will have the Jacobi metric for time-dependent systems

$$\widetilde{g}^{ij}(\boldsymbol{x},t) = \frac{1}{2m}\frac{g^{ij}(\boldsymbol{x},t)}{\mathcal{E}(t) - q^2 U(\boldsymbol{x},t)} \qquad \widetilde{g}_{ij}(\boldsymbol{x},t) = 2m\left[\mathcal{E}(t) - q^2 U(\boldsymbol{x},t)\right] g_{ij}(\boldsymbol{x},t). \tag{2.6.4}$$

Upon comparison (2.6.2) and (2.6.4) match (2.2.1.8) and (2.5.3.7) respectively. This shows that the Jacobi metric in the non-relativistic limit can be deduced from projective transformations of time-dependent systems, just as [44] demonstrates it for time-independent systems.

2.7 Application to Kepler problem

We now consider the Kepler problem of orbital motion in the presence of a central potential $U(r) = -\frac{\alpha}{r}$. Since this is a problem involving spherical symmetry, we have the spatial part of the metric as the conformally flat polar metric. We shall only consider two dimensional motion because of angular momentum conservation in a radial potential.

Thus, the Jacobi-Kepler metric is given as a conformally flat metric:

$$d\widetilde{s}^2 = (E - U(r))\left(dr^2 + r^2 d\theta^2\right) = f^2(r)\left(dr^2 + r^2 d\theta^2\right) \tag{2.7.1}$$

Here, the Gaussian curvature K_G is given by:

$$e^r = f(r)\ dr \qquad e^\theta = r f(r)\ d\theta,$$

$$de^\theta = (rf(r))'\,dr \wedge d\theta \qquad \Rightarrow \qquad {\omega^\theta}_r = \frac{(rf(r))'}{f(r)}d\theta \qquad \Rightarrow \qquad d{\omega^\theta}_r = \left(\frac{(rf(r))'}{f(r)}\right)' dr \wedge d\theta,$$

$$\therefore \qquad K_G = R^{\theta}{}_{r\theta r} = -\frac{1}{rf^2(r)}\frac{d}{dr}\left(\frac{1}{f(r)}\frac{d}{dr}\left(rf(r)\right)\right) \tag{2.7.2}$$

Thus, for $f^2(r) = E - U(r)$, the Gaussian curvature (2.7.2) in this case is given as:

$$K_G = \frac{\left(rU'(r)\right)'\left(E - U(r)\right) + r\left(U'(r)\right)^2}{2r\left(E - U(r)\right)^3} \tag{2.7.3}$$

If h is a regular value of $U(r)$ on the boundary ring, ie. $U(r) = h; x \in \partial M$ we have by continuity

$$\left(rU'(r)\right)'\left(E - U(r)\right) + r\left(U'(r)\right)^2 > 0, \qquad K_G \longrightarrow \infty \tag{2.7.4}$$

In case of the Kepler problem, we have $U(r) = -\frac{\alpha}{r}$, so the Gaussian curve K_G is:

$$K_G = -\frac{\alpha E}{2\left(rE + \alpha\right)^3}. \tag{2.7.5}$$

Thus, we can see that the curvature is classified as:

$$\forall \quad E > -\frac{\alpha}{r} \qquad \begin{cases} E < 0 & \Rightarrow \quad K_G > 0 \quad ; \quad \text{ellipse} \\ E = 0 & \Rightarrow \quad K_G = 0 \quad ; \quad \text{parabola} \\ E > 0 & \Rightarrow \quad K_G < 0 \quad ; \quad \text{hyperbola} \end{cases} \tag{2.7.6}$$

Thus, for the Kepler problem, for negative energies in the range $-\frac{\alpha}{r} < E < 0$, we will have positive curvature, and thus closed periodic orbits described by the Jacobi-Kepler metric. What motivates us to connect this theory with the Kepler problem is that it describes $\widetilde{H} = 1$ geodesic flow on $T^*S^3, K_G = 1$ energy surface.

The Hamiltonian flow along a geodesic is given by the Hamiltonian vector field operator, which for the Kepler equation, essentially becomes:

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i} \quad = \quad p_i\frac{\partial}{\partial x^i} - \alpha\frac{x^i}{r^3}\frac{\partial}{\partial p_i} \tag{2.7.7}$$

Thus, under circumstances of constant curvature, the radial equation of motion is:

$$\ddot{r} - \frac{p_\theta^2}{r^3} = -U'(r) \qquad \Rightarrow \qquad \ddot{r} = \dot{p}_r = \frac{p_\theta^2}{r^3} - U'(r). \tag{2.7.8}$$

Thus, for constant vanishing Gaussian curvature, we will have the Kepler potential, and thus, the Kepler equations of motion. However, if we consider the Jacobi metric and Hamiltonian, we will have:

$$\frac{dr}{d\sigma} = \frac{1}{E - U(r)}p_r \qquad \Rightarrow \qquad p_r = \frac{rE + \alpha}{r}\frac{dr}{d\sigma},$$

$$\frac{dp_r}{d\sigma} = \frac{dt}{d\sigma}\dot{p}_r = \frac{r}{rE + \alpha}\left(\frac{p_\theta^2}{r^3} - \frac{\alpha}{r^2}\right) \quad = \quad -\frac{\alpha}{r^2}\left(\frac{dr}{d\sigma}\right)^2 + \frac{rE + \alpha}{r}\frac{d^2r}{d\sigma^2},$$

$$\therefore \qquad \frac{d^2r}{d\sigma^2} = -\frac{Ep_\theta^2}{\left(rE + \alpha\right)^3} - \frac{\alpha}{\left(rE + \alpha\right)^2} \tag{2.7.9}$$

If one wishes to verify, it can be confirmed in (2.7.9) that:

$$\widetilde{\Gamma}^r_{jk}\frac{dx^j}{d\sigma}\frac{dx^k}{d\sigma} = \frac{Ep_\theta^2}{\left(rE + \alpha\right)^3} + \frac{\alpha}{\left(rE + \alpha\right)^2} \tag{2.7.10}$$

showing that the RHS of (2.7.9) matches that of (2.2.1.11), and our analysis is consistent.

2.7.1 Bohlin transformation and duality

The Bohlin transformation is a canonically converts the dynamics of the oscillator system into that of the Kepler system and vice versa. We shall see how the Jacobi metric for a fixed energy following a canonical transformation demonstrates this as shown in [16].

The transformation rule involves expressing the co-ordinates as a complex variable:

$$r = q_1 + iq_2 \tag{2.7.1.1}$$

The canonical transformation we shall use as shown in [30] is:

$$r \quad \longrightarrow \quad z = \frac{r^2}{2} = \left(\frac{q_1^2 - q_2^2}{2}\right) + i\left(q_1 q_2\right) = x + iy \qquad \begin{cases} x = \dfrac{q_1^2 - q_2^2}{2} \\ y = q_1 q_2 \end{cases} \tag{2.7.1.2}$$

$$x^2 + y^2 = \frac{\left(q_1^2 + q_2^2\right)^2}{4}, \quad \text{or} \quad 2\sqrt{x^2+y^2} = q_1^2 + q_2^2. \tag{2.7.1.3}$$

For the covariant momentum, in accordance with Bohlin's transformation rule:

$$p_1 = \frac{\partial x}{\partial q_1}p_x + \frac{\partial y}{\partial q_1}p_y = \ q_1 p_x + q_2 p_y, \qquad p_2 = \frac{\partial x}{\partial q_2}p_x + \frac{\partial y}{\partial q_2}p_y = -q_2 p_x + q_1 p_y,$$

$$p = p_1 + ip_2 = \left(q_1 - iq_2\right)\left(p_x + ip_y\right) \tag{2.7.1.4}$$

This transformation can also be written in matrix form as:

$$\begin{pmatrix} p_x \\ p_y \end{pmatrix} = \frac{1}{q_1^2 + q_2^2}\begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \tag{2.7.1.5}$$

Thus we obtain

$$\frac{p_1^2 + p_2^2}{q_1^2 + q_2^2} = p_x^2 + p_y^2. \tag{2.7.1.6}$$

Let $H(p,q)$ be any Hamiltonian and fix the energy E. Let us consider flow by the reparametrization $\frac{dt}{d\tau} = f(q,p)$ This immediately yields

$$\widetilde{H}(p,q) = f(p,q)(H(p,q) - E),$$

which retains the zero energy surface on the level set of H to the energy E

$$H^{-1}(E) = \{(p,q)|H(p,q) = E\}.$$

If the oscillator Hamiltonian is given as

$$H_{osc}(q_i,p_i) = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{a}{2}\left(q_1^2 + q_2^2\right) - b \tag{2.7.1.7}$$

The transformation (3.14) maps the Hamiltonian of the oscillator equation to that of Kepler,

$$H_{kepler}(x,p) = p_x^2 + p_y^2 - \frac{b}{2\sqrt{x^2+y^2}} + a, \tag{2.7.1.8}$$

thus (3.14) can be considered to be the Bohlin transformed co-ordinates and for the time being we assume $r = \sqrt{x^2+y^2} \neq 0$. This clearly yields the transformation of the oscillator Hamiltonian into the Kepler Hamiltonian.

If a and b are treated as new momenta then the null lift of the (2.7.1.8), given by

$$\widetilde{H}(x,p) = p_x^2 + p_y^2 - \frac{pz^2}{\sqrt{x^2+y^2}} + p_a^2, \tag{2.7.1.9}$$

where we have added two new conjugate variables (z, p_z), (a, p_a) and corresponding momenta being conserved. Recently, Cariglia [45] made a fine observation to connect all the energy (positive, null and negative) regimes of Kepler orbit by introducing an additional conjugate pair. This one can be done if we replace (a, p_a) pair by two additional conjugate pair (α, p_α) and (γ, p_γ) and Hamiltonian $\widetilde{H}(x,p)$ is replaced by

$$\widetilde{\mathcal{H}}(x,p) = p_x^2 + p_y^2 - \frac{pz^2}{\sqrt{x^2+y^2}} - p_\alpha^2 + p_\gamma^2.$$

2.7.2 Contact method, reparametrization and regularization

A contact form α on a $(2n+1)$-dimensional manifold M is a Pfaffian form satisfying $\alpha \wedge (d\alpha)^n \neq 0$. The contact distribution is given by $\mathcal{C}|_U = \text{ Ker } \alpha|_U$, where U is the open set in M. Given a contact form α , the Reeb vector field Z is a vector field uniquely defined by

$$i_Z\alpha = 1, \qquad i_Z d\alpha = 0. \tag{2.7.2.1}$$

Here we are interested in problem of closed Hamiltonian trajectories on a fixed energy $H = E$ surface, so we follow Weinstein's method [46]. Let P^{2n} be the total space of the principle $\mathbb{R}^*$-bundle $\pi : P \to M$, whose fibers are non-zero covectors (q,p) that vanish on the contact element $\mathcal{C}(x)$ in M. The symplectization P has a canonical 1-form α, restriction of Liouville 1-form, and the symplectic form is given by $\omega = d\alpha$. Consider the multiplicative $\mathbb{R}^*$ action on (P,ω), from the nondegeneracy of ω, there exist a unique vector field Y, called the Liouville vector field, which satisfies the following identities:

$$i_Y\omega = \alpha, \qquad \alpha(Y) = 0, \qquad L_Y\omega = \omega. \tag{2.7.2.2}$$

Since the Reeb vector field Z is a section of $\text{Ker}d\alpha|_M = 0$, hence it is proportional to $X_H|_M$. Z can be manifested as a flow of $X_H|_M$ after a time reparametrization $dt = f(q,p)\ d\tau$ introduced earlier. Thus we obtain

$$Z(x) = \frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = f(x)X_H(x), \qquad x = (q,p).$$

Claim 2.7.1. *The Reeb vector field Z is*

$$Z = \frac{X_H}{Y(H)}, \qquad \textit{where} \qquad f(x) = \frac{1}{Y(H)}. \tag{2.7.2.3}$$

Proof : By definition we know $\omega(Y,\cdot) = \alpha$ and $\alpha(Z) = 1$. Thus we obtain

$$1 = \alpha(Z) = \omega(Y, f(x)X_H) = f(x)\omega(Y,X_H) = f(x)dH(Y) = f(x)Y(H).$$

$\square$ The function $H_0 = H - E/Y(H)$ is defined on M as an invariant surface. Then the vector field $X_{H_0}|_M$ is equal to the Reeb vector field Z.

Application to Kepler equation

Consider a special symplectic transformation $(\boldsymbol{p}, \boldsymbol{q}) \to (-\boldsymbol{q}, \boldsymbol{p})$[1]. It is easy to check that this transformation leaves the symplectic form:

$$\omega = d\alpha = \sum_{i=1}^{n} dp_i \wedge dq_i = \sum_{i=1}^{n} -dq_i \wedge dp_i = \sum_{i=1}^{n} d(-q_i dp_i) = d\widetilde{\alpha}.$$

The associated Liouville vector field is $Y = \sum_{i=1}^{n} q^i \partial_{q^i}$, which satisfies $\omega(Y, \cdot) = \widetilde{\alpha}$. It is easy to check that for Kepler Hamiltonian $H = \frac{1}{2}|\boldsymbol{p}|^2 - \dfrac{\beta}{|\boldsymbol{q}|}$,

$$Y(H) = \sum_{i=1}^{n} q_i \frac{\partial H}{\partial q_i} = \sum_{i=1}^{n} (q_i)^2 \frac{\beta}{|\boldsymbol{q}|^3} = \frac{\beta}{|\boldsymbol{q}|}.$$

Thus, on isoenergetic surface we obtain

$$\frac{H-E}{Y(H)} = \frac{|\boldsymbol{q}|}{\beta}\left(\frac{1}{2}|\boldsymbol{p}|^2 - \frac{\beta}{|\boldsymbol{q}|} - E\right) = (|\boldsymbol{p}|^2 - 2E)\frac{|\boldsymbol{q}|}{\beta} - 1 = H_0.$$

Consider a smooth function

$$F = (H_0 + 1)^2/2 = \frac{(|\boldsymbol{p}|^2 - 2E)^2}{8\beta^2}|\boldsymbol{q}|^2. \tag{2.7.2.4}$$

On the fixed energy surface $H = E$, F becomes $F|_{M_E} = \frac{1}{2}$. The trajectories of the Hamiltonian flow of F on the isoenergetic surface are governed by the reparameterized time τ. The Hamiltonian vector fields of F and H_0 coincide on the level hypersurface $F = 1/2$ or equivalently $H_0 = 0$. One can easily check

$$\begin{aligned} X_F &= \frac{|\boldsymbol{q}|}{\beta} p_i \frac{\partial}{\partial q_i} - \frac{q_i}{|\boldsymbol{q}|^2}\frac{\partial}{\partial p_i} = \frac{|\boldsymbol{q}|}{\beta}\left(p_i \frac{\partial}{\partial q_i} - \frac{\beta q_i}{|\boldsymbol{q}|^3}\frac{\partial}{\partial p_i}\right) \\ &= p_i \frac{\partial}{\partial q_i} - \frac{\beta q_i}{|\boldsymbol{q}|^3}\frac{\partial}{\partial p_i} / \frac{\beta}{|\boldsymbol{q}|} = X_H / Y(H). \end{aligned}$$

Thus we establish regularization theorem due to Moser.

Theorem 2.7.1. *On the isoenergetic surface $F = 1/2$ the trajectories of the Hamiltonian flow of the function $F = \frac{(|\boldsymbol{p}|^2-2E)^2}{8\beta^2}|\boldsymbol{q}|^2$ traversed in time τ equal to trajectories of the Hamiltonian flow of the function $H = \frac{1}{2}|\boldsymbol{p}|^2 - \frac{\beta}{|\boldsymbol{q}|}$ traversed in real time t, and these two times are connected by*

$$\frac{d\tau}{dt} = \frac{\beta}{|\boldsymbol{q}|}.$$

2.7.3 Houri's canonical transformation

Another canonical transformation that can be applied to the Kepler problem, as performed by Tsuyoshi Houri in [31], involves swapping the position and momentum phase-space coordinates.

$$\widetilde{x}^i = p_i, \qquad \widetilde{p}_i = x^i. \tag{2.7.3.1}$$

[1] This transformation appears in Moser's work on regularization of Kepler orbit

Thus, the Kepler Hamiltonian will transform as:

$$H=\frac{1}{2}\left(p_1^2+p_2^2+p_3^2\right)+\frac{\alpha}{r}\quad\longrightarrow\quad\frac{1}{2}\left[\left(\widetilde{x}^1\right)^2+\left(\widetilde{x}^2\right)^2+\left(\widetilde{x}^3\right)^2\right]+\frac{\alpha}{\sqrt{\widetilde{p}_1^2+\widetilde{p}_2^2+\widetilde{p}_3^2}}. \tag{2.7.3.2}$$

As a result, if we choose a fixed energy surface $H=E$ we can further say:

$$\widetilde{H}=\left(E-\frac{\widetilde{r}^2}{2}\right)^2\left(\widetilde{p}_1^2+\widetilde{p}_2^2+\widetilde{p}_3^2\right)=\alpha^2,\qquad \widetilde{r}^2=\left(\widetilde{x}^1\right)^2+\left(\widetilde{x}^2\right)^2+\left(\widetilde{x}^3\right)^2. \tag{2.7.3.3}$$

Thus, the related metric with constant curvature $4E$ on a fixed energy surface is:

$$\widetilde{g}^{ij}(\boldsymbol{x})=\left(E-\frac{\widetilde{r}^2}{2}\right)^2\delta^{ij},\qquad \widetilde{g}_{ij}(\boldsymbol{x})=\left(E-\frac{\widetilde{r}^2}{2}\right)^{-2}\delta_{ij}. \tag{2.7.3.4}$$

So the metric is given by

$$ds^2=\left(E-\frac{|\boldsymbol{x}|^2}{2}\right)^{-2}d\boldsymbol{x}^2. \tag{2.7.3.5}$$

If we set the energy to be $E=-\dfrac{k}{2}$, we obtain

$$ds^2=4\left(k+|\boldsymbol{x}|^2\right)^{-2}d\boldsymbol{x}^2. \tag{2.7.3.6}$$

Let M_k be the space of constant curvature manifold. It is known that the Kepler phase space is geodesically incomplete, since in the collision orbits, the particle arrives to the attractive center with infinite velocity in a finite time, hence does not admit a transitive group of motion. The mapping of the inversion

$$I_k: M_k/\{0\}\to\widehat{M}_k/\{0\},$$

and $\boldsymbol{x}\to\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|^2}$ realizes isometry between its source metric g and the target metric $\widehat{g}$. Suppose

$$(I_k)_*:\boldsymbol{p}\mapsto\frac{\boldsymbol{p}}{|\boldsymbol{x}|^2}-2\frac{\boldsymbol{x}}{|\boldsymbol{x}|^4}\langle\boldsymbol{x},\boldsymbol{p}\rangle,$$

then one can easily check starting from (2.7.3.6) that

$$\begin{aligned}g_q(\boldsymbol{x}.\boldsymbol{x})=4(k+|\boldsymbol{x}|^2)^{-2}|\boldsymbol{p}|^2\mapsto&\frac{4}{\left(k+\dfrac{1}{|\boldsymbol{x}|^2}\right)^2}\frac{|\boldsymbol{p}|^2}{|\boldsymbol{x}|^4}\\&=\frac{4}{(1+k|\boldsymbol{x}|^2)^2}\langle I_*x,I_*x\rangle=\widehat{g}_{I(q)}(I_*x,I_*x),\end{aligned}$$

describing another conformally flat metric. The question that arises here is; How to connect with the Milnor construction?

If we set the energy to be $E=-\dfrac{k^2}{2}$ in (2.7.3.3), then we will have:

$$\widetilde{H}=\frac{\left(k^2+\widetilde{r}^2\right)^2}{4}\left(\widetilde{p}_1^2+\widetilde{p}_2^2+\widetilde{p}_3^2\right)=\alpha^2. \tag{2.7.3.7}$$

If we choose the reparametrization as:

$$\frac{dt}{d\tau} = \frac{r}{k}, \tag{2.7.3.8}$$

then we will have the new Hamiltonian as:

$$\mathcal{H} = \frac{r}{k}\left(H + \frac{k^2}{2}\right) = \frac{r}{k}\left(\frac{|\boldsymbol{p}|^2}{2} - \frac{\alpha}{r} + \frac{k^2}{2}\right) = \frac{r}{2k}\left(|\boldsymbol{p}|^2 + k^2\right) - \frac{\alpha}{k},$$

$$\mathbb{H} = k\mathcal{H} + \alpha = \frac{r}{2}\left(|\boldsymbol{p}|^2 + k^2\right). \tag{2.7.3.9}$$

However, Houri's approach does not preserve the form of equations of the motion or geodesic flow operator. That requires another step with Milnor's construction [14].

2.7.4 Milnor's construction

We shall now separately formulate the Kepler problem under Milnor's construction [14], which essentially involves a momentum inversion. From this formulation we shall write the metric and the trajectory equation in terms of inverse momentum.

The Kepler equation implies:

$$\frac{d\boldsymbol{p}}{dt} = -\alpha\frac{\boldsymbol{x}}{r^3} \qquad \left|\frac{d\boldsymbol{p}}{dt}\right| = \frac{\alpha}{r^2} \tag{2.7.4.1}$$

Levi-Civita showed that it is possible to simplify Kepler solutions by introducing a fictitious parameter σ such that:

$$\frac{d\sigma}{dt} = \frac{1}{r} \tag{2.7.4.2}$$

This makes the reparameterized Kepler equation of motion:

$$\frac{d\boldsymbol{p}}{d\sigma} = -\alpha\frac{\boldsymbol{x}}{r^2} = \left(E - \frac{\left|\boldsymbol{p}\right|^2}{2}\right)\frac{\boldsymbol{x}}{r} \qquad \Rightarrow \qquad \left|\frac{d\boldsymbol{p}}{d\sigma}\right| = \frac{\alpha}{r} = \frac{\left|\boldsymbol{p}\right|^2}{2} - E, \tag{2.7.4.3}$$

$$\therefore \qquad ds^2 = 4\left(2E - \left|\boldsymbol{p}\right|^2\right)^{-2}\left|d\boldsymbol{p}\right|^2. \tag{2.7.4.4}$$

Thus, there is one and only one metric on M_E that satisfies our condition. Comparing (2.7.4.4) result with the Houri's formulation (2.7.3.5), we can see that they are identical, except for a swap between momentum and co-ordinate. To describe events in the neighbourhood of infinity, we shall work with the inverted momentum co-ordinate.

$$\boldsymbol{w} = \frac{\boldsymbol{p}}{\left|\boldsymbol{p}\right|^2}, \qquad \left|\boldsymbol{w}\right|^2 = \frac{1}{\left|\boldsymbol{p}\right|^2}, \qquad 2E\left|\boldsymbol{w}\right|^2 < 1, \tag{2.7.4.5}$$

$$\therefore \quad \boldsymbol{p} = \frac{\boldsymbol{w}}{\left|\boldsymbol{w}\right|^2} \qquad \Rightarrow \qquad d\boldsymbol{p} = \frac{d\boldsymbol{w}}{\left|\boldsymbol{w}\right|^2} - 2\frac{\left(\boldsymbol{w}.d\boldsymbol{w}\right)\boldsymbol{w}}{\left|\boldsymbol{w}\right|^4}, \qquad \left|d\boldsymbol{p}\right|^2 = \frac{\left|d\boldsymbol{w}\right|^2}{\left|\boldsymbol{w}\right|^4} \tag{2.7.4.6}$$

Using (2.7.4.3), (2.7.4.5) and (2.7.4.6) and defining $(\ )' = \frac{d}{d\sigma}$, we will get:

$$\boldsymbol{p}' = \left(E - \frac{\left|\boldsymbol{p}\right|^2}{2}\right)\frac{\boldsymbol{x}}{r} = \frac{2E\left|\boldsymbol{w}\right|^2 - 1}{2\left|\boldsymbol{w}\right|^2}\frac{\boldsymbol{x}}{r},$$

$$\Rightarrow \qquad \frac{\boldsymbol{w}'}{|\boldsymbol{w}|^2} - 2\frac{(\boldsymbol{w}.\boldsymbol{w}')\,\boldsymbol{w}}{|\boldsymbol{w}|^4} = \frac{2E|\boldsymbol{w}|^2 - 1}{2|\boldsymbol{w}|^2}\frac{\boldsymbol{x}}{r} \tag{2.7.4.7}$$

$$\text{and} \qquad \frac{|\boldsymbol{w}'|^2}{|\boldsymbol{w}|^4} = \left(\frac{2E|\boldsymbol{w}|^2 - 1}{2|\boldsymbol{w}|^2}\right)^2 \qquad \Rightarrow \qquad 4|\boldsymbol{w}'|^2 = \left(2E|\boldsymbol{w}|^2 - 1\right)^2 \tag{2.7.4.8}$$

If we now substitute the fixed energy level $E = -\frac{k^2}{2}$ in (2.7.4.8), then we will have the metric in terms of the inverse momentum given as:

$$ds^2 = 4\left(1 + k^2|\boldsymbol{w}|^2\right)^{-2}|d\boldsymbol{w}|^2 \tag{2.7.4.9}$$

which is the inverse-momentum version of (2.7.4.4) and a constant mean-curvature metric. From (2.7.4.7), we get the trajectory equation in terms of inverse momentum as:

$$\boldsymbol{x} = \frac{|\boldsymbol{w}|^2\boldsymbol{w}' - 2\,(\boldsymbol{w}.\boldsymbol{w}')\,\boldsymbol{w}}{|\boldsymbol{w}|^2\left(2E|\boldsymbol{w}|^2 - 1\right)}r = 2\alpha\frac{2\,(\boldsymbol{w}.\boldsymbol{w}')\,\boldsymbol{w} - |\boldsymbol{w}|^2\boldsymbol{w}'}{\left(1 - 2E|\boldsymbol{w}|^2\right)^2} \tag{2.7.4.10}$$

Thus, $\boldsymbol{x}$ can be expressed as a smooth function of the parameter σ. If we use t in place of σ, the function stops being smooth only at the point $\boldsymbol{x} = 0$.

2.7.5 Geodesic flow

Now we will see if the form of geodesic flow is preserved after using momentum inversion upon Houri's canonical transformation. The Hamiltonian (2.7.3.7) describing geodesics on such spaces under a momentum inversion for $E = -\frac{k}{2}$ [47] is given by

$$\widetilde{H} = \frac{1}{4}\left(1 + k|\boldsymbol{x}|^2\right)^2|\boldsymbol{p}|^2. \tag{2.7.5.1}$$

From this Hamiltonian, setting we can derive the Hamiltonian flow vector field

$$X_{\widetilde{H}} = \frac{\partial \widetilde{H}}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial \widetilde{H}}{\partial x^i}\frac{\partial}{\partial p_i} \quad = \quad 2\widetilde{H}^{\frac{1}{2}}|\boldsymbol{p}|\left[\frac{p^i}{|\boldsymbol{p}|^3}\frac{\partial}{\partial x^i} - kx_i\frac{\partial}{\partial p_i}\right].$$

Thus we finally obtain

$$\therefore \qquad \left(2\widetilde{H}^{\frac{1}{2}}|\boldsymbol{p}|\right)^{-1}X_{\widetilde{H}} = \frac{p^i}{|\boldsymbol{p}|^3}\frac{\partial}{\partial x^i} - x_i\frac{\partial}{\partial p_i}. \tag{2.7.5.2}$$

Comparing the flow operator above with the geodesic flow in (2.7.7), we obtain the quasi-Hamiltonian vector field of Kepler equation in momentum space

$$X^{mom}_{Kepler} = \left(2k^2\widetilde{H}^{\frac{1}{2}}|\boldsymbol{p}|\right)^{-1}X_{\widetilde{H}}. \tag{2.7.5.3}$$

Thus, we can see that combining Houri's transformation with Milnor's momentum inversion preserves the form of the geodesic flow, aside from a momentum factor. A vector field X on a symplectic manifold (M,ω) is quasi-Hamiltonian if there exists a (nowhere-vanishing) function Λ such that X is a Hamiltonian vector field $\Lambda X \in \mathcal{X}_H(M)$, thus $i_{\Lambda X}\omega = dH$. This condition can alternatively be written as as $i_X(\Lambda\omega) = dH$, but the point is that the 2-form $\Lambda\omega$ is not closed in the general case.

By applying the special canonical transformation that interchanges $\boldsymbol{x}$ and $\boldsymbol{p}$, the Kepler equation on momentum space transforms to the usual Kepler equation with the Hamiltonian

$$H = \frac{1}{4}\left(k + |\boldsymbol{p}|^2\right)^2 |\boldsymbol{x}|^2.$$

Finally, we will explore the results of parameterizing the JM metric and Kepler equation with the eccentric anomaly.

2.7.6 JM metric and Kepler equation parameterized by eccentric anomaly

The Hamiltons equations of motion for the Kepler Hamiltonian under Houri's canonical transformation (2.7.3.3) are:

$$\begin{aligned}
\dot{\widetilde{x}}^i &= \frac{\partial \widetilde{H}}{\partial \widetilde{p}_i} = 2\left[E - \frac{1}{2}\sum_{n=1}^{3}(\widetilde{x}^n)^2\right]^2 \widetilde{p}_i, \\
\dot{\widetilde{p}}_i &= \frac{\partial \widetilde{H}}{\partial \widetilde{x}^i} = 2\left[E - \frac{1}{2}\sum_{n=1}^{3}(\widetilde{x}^n)^2\right]\left(\sum_{n=1}^{3}\widetilde{p}_n^2\right)\widetilde{x}^i.
\end{aligned} \tag{2.7.6.1}$$

To proceed to equations of motion, we shall use (2.7.6.1) to write:

$$\begin{aligned}
\ddot{\widetilde{x}}^i &= -2\left\{E - \frac{1}{2}\sum_{n=1}^{3}(\widetilde{x}^n)^2\right\}^{-1}\left(\sum_{k=1}^{3}\widetilde{x}^k\dot{\widetilde{x}}^k\right)\dot{\widetilde{x}}^i + 4\left\{E - \frac{1}{2}\sum_{n=1}^{3}(\widetilde{x}^n)^2\right\}\widetilde{H}\widetilde{x}^i \\
&= -2\frac{\left(\widetilde{\boldsymbol{x}}\cdot\dot{\widetilde{\boldsymbol{x}}}\right)\cdot\dot{\widetilde{\boldsymbol{x}}}}{\Lambda} + 4\Lambda\widetilde{H}\widetilde{\boldsymbol{x}},
\end{aligned}$$

where $\Lambda = (E - \frac{1}{2}\sum_{n=1}^{3}(\widetilde{x}^n)^2)$. Let us write $\widetilde{\boldsymbol{x}}$ as $\boldsymbol{x}$, hence we obtain

$$\ddot{\boldsymbol{x}} = -2\frac{(\boldsymbol{x}\cdot\dot{\boldsymbol{x}})\cdot\dot{\boldsymbol{x}}}{\Lambda} + 4\Lambda\widetilde{H}\boldsymbol{x}. \tag{2.7.6.2}$$

It is known that the Laplace Lenz Runge vector

$$A(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \frac{1}{\mu}\left(2H + \frac{\mu}{|\boldsymbol{x}|}\right)\boldsymbol{x} - \frac{1}{\mu}(\boldsymbol{x}\cdot\dot{\boldsymbol{x}})\dot{\boldsymbol{x}} \tag{2.7.6.3}$$

is a conserved quantity for the Kepler flow, we can re-write this equation using $A(\boldsymbol{x}, \dot{\boldsymbol{x}})$. Using the Laplace Lenz Runge vector we obtain

$$\frac{2\mu}{\Lambda}\left(A(\boldsymbol{x}, \dot{\boldsymbol{x}}) - \frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) = -2\frac{(\boldsymbol{x}\cdot\dot{\boldsymbol{x}})\cdot\dot{\boldsymbol{x}}}{\Lambda} + 4\Lambda\widetilde{H}\boldsymbol{x},$$

where $H = E = \widetilde{H}\Lambda^2$. Thus equation (2.7.6.2) can be written as

$$\ddot{\boldsymbol{x}} + \frac{2\mu}{\Lambda}\frac{\boldsymbol{x}}{|\boldsymbol{x}|} = \frac{2\mu A}{\Lambda}. \tag{2.7.6.4}$$

Kepler equation parameterizing the eccentric anomaly

An advantage of the eccentric anomaly is that it is well suited to describe Kepler motion in position space. Therefore we derive the equation of motion w.r.t. this parameter.

Let us reparameterize the time as

$$dt = \frac{|\boldsymbol{x}|}{\epsilon} ds. \tag{2.7.6.5}$$

Thus we obtain

$$\frac{d\boldsymbol{x}}{ds} = \frac{d\boldsymbol{x}}{dt}\frac{dt}{ds} = \dot{\boldsymbol{x}}\frac{|\boldsymbol{x}|}{\epsilon}.$$

The second derivative yields

$$\frac{d^2\boldsymbol{x}}{ds^2} = \frac{1}{|\boldsymbol{x}|^2}\left(\boldsymbol{x}\cdot\frac{d\boldsymbol{x}}{ds}\right)\frac{d\boldsymbol{x}}{ds} + \frac{|\boldsymbol{x}|^2}{\epsilon^2}\ddot{\boldsymbol{x}} = \frac{1}{|\boldsymbol{x}|^2}\left(\boldsymbol{x}\cdot\frac{d\boldsymbol{x}}{ds}\right)\frac{d\boldsymbol{x}}{ds} - \frac{\mu}{\epsilon^2|\boldsymbol{x}|}\boldsymbol{x},$$

where we have used the Kepler equation $\ddot{\boldsymbol{x}} = -\mu\frac{\boldsymbol{x}}{|\boldsymbol{x}|^3}$, and the Laplace-Runge-Lenz vector is given by:

$$\begin{aligned} A(\boldsymbol{x},\dot{\boldsymbol{x}}) &= \frac{1}{\mu}\left(2E + \frac{\mu}{|\boldsymbol{x}|}\right)\boldsymbol{x} - \frac{1}{\mu}\left(\boldsymbol{x}\cdot\dot{\boldsymbol{x}}\right)\dot{\boldsymbol{x}} \\ &= -\frac{\epsilon^2}{\mu}\left[\frac{1}{|\boldsymbol{x}|^2}\left(\boldsymbol{x}\cdot\frac{d\boldsymbol{x}}{ds}\right)\frac{d\boldsymbol{x}}{ds} - \frac{\mu}{\epsilon^2}\frac{\boldsymbol{x}}{|\boldsymbol{x}|} + \boldsymbol{x}\right], \qquad \epsilon^2 = -2E. \end{aligned}$$

Here we consider the case of negative energy, ie. bounded orbits. Therefore we obtain

$$\frac{d^2\boldsymbol{x}}{ds^2} + \boldsymbol{x} = -\frac{\mu}{\epsilon^2}\boldsymbol{A}, \tag{2.7.6.6}$$

which is the form of a perturbed oscillator. Let us start with the Hamiltonian

$$\widetilde{H} = \alpha^2 = \left(E - \frac{\widetilde{r}^2}{2}\right)^2\left(\sum_{n=1}^{3}\widetilde{p}_n^2\right) = \frac{1}{4}\left(\epsilon^2 + |\boldsymbol{x}|^2\right)^2|\boldsymbol{p}|^2,$$

where we have used $2E = -\epsilon^2$. Now define

$$G(\boldsymbol{x},\boldsymbol{p}) = \widetilde{H}^{1/2} = \frac{1}{2}\left(\epsilon^2 + |\boldsymbol{x}|^2\right)|\boldsymbol{p}|.$$

We now consider regularized Kepler Hamiltonian system. The system of the Hamiltonian obtained from

$$\widetilde{G}(\boldsymbol{x},\boldsymbol{p}) = \frac{1}{2\epsilon}\left(\epsilon^2 + |\boldsymbol{x}|^2\right)|\boldsymbol{p}| - \frac{\mu}{\epsilon}, \qquad \epsilon \neq 0, \tag{2.7.6.7}$$

is given by

$$\dot{\boldsymbol{p}} = \frac{|\boldsymbol{p}|}{\epsilon}\boldsymbol{x}, \qquad \dot{\boldsymbol{x}} = -\frac{1}{|\boldsymbol{p}|^2}\left(\widetilde{G}(\boldsymbol{x},\boldsymbol{p}) + \frac{\mu}{\epsilon}\right).$$

By the first equation, $\boldsymbol{x} = \frac{\epsilon}{|\boldsymbol{p}|}\dot{\boldsymbol{p}}$, we obtain

$$\ddot{\boldsymbol{p}} = \frac{1}{|\boldsymbol{p}|^2}(\boldsymbol{p}\cdot\dot{\boldsymbol{p}})\dot{\boldsymbol{p}} - \frac{1}{\epsilon|\boldsymbol{p}|}\left(\widetilde{G}(\boldsymbol{x},\boldsymbol{p}) + \frac{\mu}{\epsilon}\right)\boldsymbol{p}.$$

Its restriction to the level set $\left[(\boldsymbol{x},\boldsymbol{p})|\widetilde{G}(\boldsymbol{x},\boldsymbol{p}) = 0\right]$ is flow of the Kepler problem in the momentum space parameterized by the eccentric anomaly.

Chapter 3

Instantons and self-dual systems on curved spaces

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3.1 Introduction

An instanton or pseudo-particle is a concept in mathematical physics that describes solutions to equations of motion of classical field theory on a Euclidean space-time. The first such solutions discovered were found to be localized in space-time, hence, the name instanton or pseudoparticle. Instantons are important in quantum field theory because:

(a) They are leading quantum corrections to classical motion equations in the path integral

(b) They are useful for studying tunneling behaviour in systems like the Yang-Mills theory

Gravity and supergravity, like Yang-Mills theory, are gauge theories, implying similar roles for their respective instantons. Instantons (for a set of self-contained lectures look for [48]) are non singular solutions of classical equations in 4-dimensional Euclidean space, and a useful tool to study low-dimensional sigma models and supersymmetric QCD. Since instantons are non-perturbative objects, they play an important role in defining the vacuum structure of QCD. It was found by Belavin, Polyakov, Schwarz and Tyupkin [49], which is why it is known as BPST instantons in literature. Their role is to:

1. provide stationary phase points in path integrals for amplitude to tunnel between topologically distinct field configurations [50]

2. possibly play a role in quark confinement [51, 52] (for more details see [53, 54, 55]) that lead baryons to decay into leptons and asymptotic freedom of QCD

3. contribute to the anomalous divergence of the axial vector current [56]

Gravitational instantons are non singular complete positive definite metrics that satisfy the classical vacuum Einstein equations or the Einstein equations with Λ [57].

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}.$$

In Euclidean quantum gravity, they are stationary phase metrics in the path integrals for the partition functions Z [50] of the thermal and volume canonical ensembles [57, 58]. In these cases the instanton action dominates the contribution to $-\log Z$. This action is related to the areas of the bolts and to the nut charges and potentials. Nuts and bolts exhibit a symmetry analogous to duality invariance in electromagnetism. Bolts are analogous to "electric" type mass monopoles and nuts to gravitational dyons with a real electric type mass-monopole and an imaginary "magnetic" type mass-monopole. The appearance of magnetic monopole induces Dirac string-like singularity into the metric which can be further removed by appropriate identifications and changes in the topology of the four manifold. So Nuts cannot occur in the classical regimes without some quantum fluctuations of the background contrary to the appearance of bolts. This implies that the bolts have an intrinsic gravitational entropy equal to one quarter the sum of their areas. This generalises the results obtained for black holes and cosmological event horizons [58, 59, 60].

Gravitational instantons, like gauge instantons, have been historically studied to describe the non-perturbative transitions in quantum gravity, and by analytic continuation producing real-time gravitational backgrounds [50, 61, 62]. Dealing with (anti)-self-dual systems makes the task of solving Einsteins equations substantially easier, producing first order equations like the (anti)-self-dual Yang-Mills equations, often related to interesting integrable systems [63]. Since the late 70s, many systems were developed from reductions of self-dual Yang-Mills system, hinting that all integrable systems can be obtained from similar such reductions [63, 64, 65], the generalized Darboux-Halphen system being one such system.

The chapter is organized as follows: we begin in section 3.2 with an exercise in the study of the Euclidean Schwarzschild instanton, starting with a review of the standard results of the bottom-up formulation of emergent gravity [66], and introduce the euclidean Schwarzschild solution, with a wise choice for the Darboux coordinates in which we write the corresponding metric. Then we obtain the set of symplectic $U(1)$ gauge fields and derive the corresponding vector fields and check the Jacobi identity for the Poisson and Lie algebra. Next we realize the Seiberg Witten map between ordinary and NC gauge fields and find that the solution is neither self-dual nor anti self-dual. In the next section, from the set of tetrads we obtain the spin-connections and the curvature components and from that we get the Ricci tensor and obtain Ricci flatness for the metric. Ricci flatness condition also translates into a vacuum solution. In the penultimate section, we compute the bulk and boundary contribution to the topological invariants namely Euler characteristics and the Hirzebruch signature complex. Here we also obtain $SU(2)_\pm$ gauge fields for emergent Schwarzschild instanton and reconfirm the fact that both the gauge fields make an equal contribution to the overall Euler invariant or the signature. Thus emergent Scwarzschild solution can be seen as the sum of $SU(2)_L$ instantons and $SU(2)_R$ anti-instantons, thus explaining the generic feature of stability for a Ricci-flat manifold like the one we dealt with. We conclude with some comments and future directions. The appendices contain some details of the computations, namely some identities from differential geometry that has been used, also the t'Hooft matrices and the full set of $SU(2)_\pm$ gauge fields in matrix notation.

In section 3.3, the gDH system is introduced and a constrained system is derived from it. Then the solutions of both the gDH and the constrained systems are discussed. We derive following [67], the gDH system from a ninth-order dynamical system that is obtained as a reduction of the SDYM field equations. We provide some details in our derivation that were not included in earlier papers. Then we discuss the constrained system in the framework of a fifth-order system that arise as a special case of the SDYM reduction. Then we formulate a Lax pair and a Hamiltonian for the reduced system introduced.

In section 3.4, we will study the Bianchi-IX Euclidean metrics, starting by performing a geometric analysis of the Bianchi-IX metric, directly exploring both connection-wise and curvature-wise self-dual cases, bringing us to the Euler-Top and classical Darboux-Halphen cases respectively, followed by computation of the general form of the curvature components. A brief note will discuss how the classical case arises from the generalized one, and also explore why we cannot always find a metric that gives rise to the generalized system. We then proceed to derivations and discussions of the related Ricci flow equations related to the system. We will then explore how the Chazy equation emerges from the classical Darboux-Halphen system, a result of curvature-wise self-duality, as well how others like the Ramanujan and Ramamani systems are related to it. This will be followed up by a detailed analysis of integrability of the Bianchi-IX to see if self-duality implies integrability.

Finally, in section 3.5, we study the Taub-NUT, a special case of the Bianchi-IX as another integrable system, the Bertrand space-time with magnetic field, starting with preliminaries on mechanical systems with magnetic field interactions, then compute first-integrals similar to the angular momentum and the Laplace-Runge-Lenz vector, in forms specifically for the Taub-NUT. We deduce such first-integrals using equations of motion and analytically using a momentum polynomial expansion. We then proceed to compare Taub-NUT metric to Euclidean Bertrand space-time with magnetic monopoles and dipoles. Demonstrating such a similarity allows the intensely studied Bertrand space-times to share many important properties, and conversely extend properties of the Taub-NUT to Bertrand spaces with magnetic fields. This helps us identify symmetries and conserved quantities of Taub NUT and employ its curvature properties for Bertrand space-times. Afterwards, we study the conserved quantity called the Fradkin tensor under Bohlin-Arnold-Vassiliev transformation which are bound to have such Killing tensors embedded. Next, we derive the Taub-NUT from a special case of self-dual Bianchi-IX metric described by the classical Darboux-Halphen system. Then we geometrically analyze it, computing curvature and confirming its self-duality as a gravitational instanton.

$$R_{\mu\nu\rho\sigma} = \pm\frac{1}{2}\varepsilon_{\mu\nu}{}^{\lambda\gamma}R_{\lambda\gamma\rho\sigma} \tag{3.1.1}$$

This analysis helps us explore the metric as an integrable system, and also to compute topological invariants shared with comparable Bertrand space-times with magnetic fields. After a short introduction to Killing Stäckel tensor and Yano tensors, we will focus on the latter. After a brief overview of their properties, we will attempt to find them embedded within conserved quantities. Then, we see if it exhibits a graded Lie-algebra structure that decides if higher order Killing-Yano tensors can be constructed from it. Finally, we derive hyperkähler structures of the Taub-NUT, and compare them to the Killing-Yano tensors to see if they also exhibit quaternionic algebra.

3.2 Schwarzschild instanton in Emergent Gravity

The Euclidean Schwarzschild solution is a canonical example of a gravitational instanton exhibiting one parameter continuous symmetry group as opposed to two parameter continuous symmetry group exhibited by almost all other known gravitational instantons. It is an well known asymptotically flat (AF) gravitational instanton alongside the Euclidean Kerr and flat space $S \times \mathbb{R}^3$ which is a trivial examle representing the class [68]. Flat space E^4 is also known to be the unique asymptotically Euclidean gravitational instanton. It was an unproven conjecture that Euclidean Schwarzschild and Kerr are the only non-trivial AF gravitational instantons besides flat space, due to some blackhole uniqueness theorems which was later proven to be false by Chen and Teo [69].

It is worth paying attention to the fact that the thermal nature of black hole emission can be related directly to the properties of Euclidean Schwarzschild solution ala Hawking. In the Euclidean approach to quantum field theory one attempts to define quantities on a "Euclidean subsection" and then obtain the physical space-time quantities by analytic continuation. Particularly, the Feynman propagator for a field on space-time is obtained by analytically continuing the Green's function on Euclidean subsection. Thus one is naturally led to study and examine the salient features of Euclidean Schwarzschild solution.

Mathematically the Euclidean Schwarzschild 4-manifold M is a complete solution to the Euclidean Einstein's equations with zero cosmological constant Λ, and has the non-trivial topology $M \cong \mathbb{R}^2 \times S^2$. In other words it is a Ricci flat manifold. It is not a self-dual solution (e.g. the Taub-NUT metric or the Eguchi- Hanson metric) although classified as an AF type gravitational instaton. We have a particularly nice form of the metric g on a dense open subset $(\mathbb{R}^2 \setminus \{O\}) \times S^2 \subset M \cong \mathbb{R}^2 \times S^2$ of the Euclidean Schwarzschild manifold. It is convenient to use polar coordinates (r, τ) on $\mathbb{R}^2 \setminus \{O\}$ in the range $r \in (2m, \infty)$ and $\tau \in [0, 8\pi m)$, where $m > 0$ is a fixed constant related to the mass of the black hole.

The metric then takes the form on the open, dense coordinate chart $U := (\mathbb{R}^2 \setminus \{O\}) \times (S^2 \setminus (\{S\} \cup \{N\})) \subset M \cong \mathbb{R}^2 \times S^2$:

$$ds^2 = \left(1 - \frac{2m}{r}\right) d\tau^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\Theta^2 + \sin^2\Theta \; d\phi^2),$$

where $d\Omega^2 = d\Theta^2 + \sin^2\Theta \; d\phi^2$ is the line element projection onto the unit sphere S^2 in spherical coordinates, $\Theta \in (0, \pi)$ and $\phi \in [0, 2\pi)$, and on the open coordinate chart $(S^2 \setminus (\{S\} \cup \{N\})) \subset S^2$. Despite the apparent singularity of the metric at the origin $O \in \mathbb{R}^2$, it can be extended analytically to the whole $\mathbb{R}^2 \times S^2$ as demonstrated in Wald [70]. The $U(1)$ action defined by $\tau \mapsto \tau + 4m\lambda$ for $e^{i\lambda} \in U(1)$ leaves this metric invariant, and thus defines the Killing vector field

$$X := \frac{1}{4m}\frac{\partial}{\partial\tau},$$

which (together with the $U(1)$ action itself) clearly extends to a Killing field on the whole Euclidean Schwarzschild manifold, which we will denote by X. Now consider the differential 1-form $\xi := g(X, \cdot\,)$ dual to X. In our coordinate chart U it takes the form

$$\xi = \frac{1}{4m}\left(1 - \frac{2m}{r}\right) d\tau.$$

General considerations about Killing's equations on a Ricci flat manifold yield that $d\xi$ is a harmonic 2-form, which on a complete manifold is equivalent to saying that it is closed and co-closed and thus harmonic.

The correspondence between noncommutative (NC) U(1) gauge theory and gravity has gained much attention in the context of emergent gravity [71, 72, 73, 74]. Current research in the field of instantons [75, 76] reveals that gravitational instantons in Einstein gravity are equivalent to $U(1)$ instantons in NC gauge theory. In other words, self-dual electromagnetism on NC space-time is equivalent to self-dual Einstein gravity [77]. This implies that gravity can emerge from electromagnetism defined in NC space-time.

The relation between Yang-Mills instantons and gravitational instantons are further understood in [78] where it was shown that all gravitational insatantons are $SU(2)$ Yang-Mills instantons on a Ricci-flat 4- manifold but the reverse is not necessarily true. Gravitational instantons satisfy the same self-dual equations of $SU(2)$ Yang-Mills instantons. The gravitational instanton which is a solution of (anti) self-dual gravity emerges either from $SU(2)_L$ or $SU(2)_R$ Yang-Mills instanton sector, and the corresponding gauge fields constructed from Yang-Mills instanton generate (anti) self-dual gravity. In [78] the result was further extended to include general Einstein manifolds [79]: all Einstein manifolds with or without cosmological constant are Yang-Mills instantons in $O(4) = SU(2)_L \times SU(2)_R$ gauge theory but the reverse is not true. In fact they arise as a sum of instantons coming both from $SU(2)_L$ instanton and $SU(2)_R$ anti-instanton. This may explain the stability of the four dimensional Einstein manifold compared to the five dimensional Kaluza-Klein vacuum.

Here, we deal with a specific example of a Ricci flat Einstein manifold: the Euclidean Schwarzschild black hole. It was discussed in [78] that the Euclidean solution outside of the (anti)self-dual gravity is a combination of both $SU(2)_L$ and $SU(2)_R$ Yang-Mills instanton. Following the bottom-up approach of Emergent Gravity [66], we construct vector fields from the Euclidean Schwarzschild instanton and calculate the equations of motion and Jacobi identity. Using the Seiberg-Witten map we find the symplectic field strength and check the absence of self-duality for Euclidean Schwarzschild. We explicitly show the Ricci flatness and shed light on the vacuum Einstein solution as is evident from the energy momentum tensor that can be computed exactly exploiting the relation between spin connections and structure constants for the Schwarzschild solution. We further study their geometric properties by calculating the topological invariants of the $U(1)$ gauge fields [80] derived from emergent Schwarzschild metric.

3.2.1 Review of bottom-up Emergent Gravity formulation

The mathematical tool to quantize the dynamical system [81] is to specify the Poisson structure θ such that

$$\theta = \frac{1}{2}\sum_{A,B=1}^{N} \theta^{AB}\frac{\partial}{\partial x^A} \wedge \frac{\partial}{\partial x^B} \in \Gamma(\wedge^2 TM),$$

and then the differentiable manifold M endowed with θ describes a Poisson manifold (M, θ). The Poisson structure defines an $\mathbf{R}$-bilinear antisymmetric operation $\{,\}_\theta$: $C^\infty(M)\times C^\infty(M) \to C^\infty(M)$

$$(f, g) \mapsto \{f, g\}_\theta = \langle \theta, df \otimes dg\rangle = \theta^{AB}(x)\frac{\partial f(x)}{\partial x^A}\frac{\partial g(x)}{\partial x^B},$$

and the Poisson bracket satisfy the Leibniz rule and Jacobi identity as follows:

$$\begin{aligned}&\{f,gh\}_\theta = g\{f,h\}_\theta + \{f,g\}_\theta h,\\ &\{f,\{g,h\}_\theta\}_\theta + \{g,\{h,f\}_\theta\}_\theta + \{h,\{f,g\}_\theta\}_\theta = 0,\end{aligned} \qquad \forall\, f,g,h \in C^\infty(M),$$

where the Poisson structure θ reduces to symplectic structure when it is nondegerate.

Co-ordinates defined on a phase-space are known to be noncommutative (NC) under Poisson-Bracket (PB) operations. This would imply that the phase-space employed in classical mechanics is a sort of NC space, where the position $\boldsymbol{x}$ and momenta $\boldsymbol{p}$ are noncommutative variable pairs exhibit the PB relations:

$$\left\{x^i, x^j\right\} = \{p_i, p_j\} = 0, \qquad \left\{x^i, p_j\right\} = \delta^i_j. \tag{3.2.1.1}$$

NC spaces with position variables defined as $\boldsymbol{x}$ and product operations re-defined as the Weyl-Moyal products are characterised by the PB relation:

$$\left\{z^i, z^j\right\} = \theta^{ij}. \tag{3.2.1.2}$$

However, in case of phase-spaces, when symplectic $U(1)$ gauge fields are involved, the gauge-covariant momenta have the PB relation

$$\{\Pi_i, \Pi_j\}_\theta = F_{ij} = \partial_i A_j - \partial_j A_i, \qquad \Pi_i = p_i - A_i, \tag{3.2.1.3}$$

while the NC spaces exhibit the PB relation

$$\left\{X^i, X^j\right\}_\theta = \theta^{ij} - \theta^{im}\widehat{F}_{mn}\theta^{nj}, \qquad \widehat{F}_{ij} = \partial_i A_j - \partial_j A_i + \{A_i, A_j\}_\theta\,.$$

We shall demonstrate how to set up a noncommutative space. To this end, we shall start with a few preliminaries, before proceeding to the construction of NC space employed in the study of Emergent Gravity.

Noncommutative spaces and the Weyl-Moyal product

We will introduce and deal with the aspect of noncommutative geometry, via the **Weyl-Moyal** $\star$-product. This is at the heart of our theory for creation of various $U(1)$ gauge fields we shall deal with as elaborated in [82], and requires our attention.

In regular commutative geometry, the usual co-ordinate commutator relation is:

$$\left[x^i, x^j\right]_. = 0 \qquad \Rightarrow \qquad x^i.x^j = x^j.x^i. \tag{3.2.1.4}$$

However, we shall now redefine the product operation as:

$$f(x).g(x) \longrightarrow f(x)\star g(x) = \left[\exp\left(\frac{i}{2}\theta^{ij}\partial_{x^i}\partial_{y^j}\right) f(x)g(y)\right]_{y=x}. \tag{3.2.1.5}$$

The result is that the co-ordinate $\star$-product (for $f(x) = x^i, g(x) = x^j$) is now given as

$$x^i \star x^j = \left[\exp\left(i\theta^{ij}\partial_{x^i}\partial_{y^j}\right)x^i y^j\right]_{x=y} = x^i.x^j + \frac{i}{2}\theta^{ij} \neq \pm\left(x^i.x^j + \frac{i}{2}\theta^{ji}\right) = \pm\left(x^j \star x^i\right),$$

$$\therefore \qquad \left[x^i, x^j\right]_\star = x^i \star x^j - x^j \star x^i = i\left\{x^i, x^j\right\}_\theta = i\theta^{ij}. \tag{3.2.1.6}$$

Thus, under the new product rule, the co-ordinates do not commute. One must keep in mind that this does not necessarily imply that they anti-commute either, as has been shown.

The Seiberg-Witten map

Now we shall consider a deformation of the co-ordinate via a map, formulated by Nathan Seiberg, and Edward Witten [83], known as the Seiberg-Witten map given by:

$$f_{SW} : M \to N : x^i \quad \longrightarrow \quad X^i = x^i + \theta^{im} \widehat{A}_m, \tag{3.2.1.7}$$

similar to how gauge-covariant momenta Π_i is defined in (3.2.1.3). The $\star$-commutator of these co-ordinates is given by:

$$\begin{aligned}
\left[X^i, X^j\right]_\star &= i\left\{X^i, X^j\right\}_\theta, \\
\left\{X^i, X^j\right\}_\theta &= \left\{x^i, x^j\right\}_\theta + \theta^{jn}\left\{x^i, \widehat{A}_n\right\}_\theta + \theta^{im}\left\{\widehat{A}_m, x^j\right\}_\theta + \theta^{im}\theta^{jn}\left\{\widehat{A}_m, \widehat{A}_n\right\}_\theta \\
&= \left\{x^i, x^j\right\}_\theta - \left\{x^i, \widehat{A}_n\right\}_\theta \theta^{nj} + \theta^{im}\left\{\widehat{A}_m, x^j\right\}_\theta - i\,\theta^{im}\left[\widehat{A}_m, \widehat{A}_n\right]_\star \theta^{nj} \\
&= \theta^{ij} - \theta^{im}\partial_m\widehat{A}_n\theta^{nj} + \theta^{im}\partial_n\widehat{A}_m\theta^{nj} - i\,\theta^{im}\left[\widehat{A}_m, \widehat{A}_n\right]_\star \theta^{nj}, \\
\therefore \qquad \left[X^i, X^j\right]_\star &= i\theta^{ij} - i\theta^{im}\Big(\partial_m\widehat{A}_n - \partial_n\widehat{A}_m - i\big[\widehat{A}_m, \widehat{A}_n\big]_\star\Big)\theta^{nj}.
\end{aligned} \tag{3.2.1.8}$$

Now we shall define the noncommutative field strength in the usual noncommutative frame as

$$\widehat{F}_{mn} = \partial_m\widehat{A}_n - \partial_j\widehat{A}_m - i\left[\widehat{A}_m, \widehat{A}_n\right]_\star = \partial_m\widehat{A}_n - \partial_j\widehat{A}_m + \left\{\widehat{A}_m, \widehat{A}_n\right\}_\theta. \tag{3.2.1.9}$$

For the LHS of (3.2.1.8), we can say that the Seiberg-Witten map (3.2.1.7) is a diffeomorphic map $f_{SW} : M \to N$ such that

$$\left\{X^i, X^j\right\}_\theta = \Theta^{ij} = (B+F)^{-1}. \tag{3.2.1.10}$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$ is the gauge field strength defined on commutative space. For the LHS, we can further say

$$(B+F)^{-1} = \left[B\left(1+\theta F\right)\right]^{-1} = \left(1+\theta F\right)^{-1}\theta \equiv \theta - \theta F\left(1+\theta F\right)^{-1}\theta,$$

$$\therefore \qquad \theta - \theta F\left(1+\theta F\right)^{-1}\theta = \theta - \theta\widehat{F}\theta \qquad \Rightarrow \qquad \widehat{F} = F\left(1+\theta F\right)^{-1} \tag{3.2.1.11}$$

This concludes the setup of NC spaces and the formulation of NC $U(1)$ gauge fields in terms of dynamical $U(1)$ gauge fields.

The application of Darboux theorem or Moser lemma[81] of symplectic geometry to electromagnetism defined on the symplectic space gives rise to an equivalence principle. An arbitrary deformation of symplectic deformation can not be distinguishable locally from canonical form. The electromagnetism on symplectic space-time can be a theory of gravity[84]: Starting with symplectic form $\omega_0 = B$, the deformation of ω_0 generate dynamical gauge fields such that $\omega_1 = B + F$, where $F = dA$. It is always possible to eliminate F by a suitable coordinate transformation as far as the 2-form B is closed and nondegenerate because in this case the gauge symmetry becomes a space-time symmetry rather than an internal symmetry. This very fact indeed paves the way for a connection between NC gauge fields and space-time geometry.

For a given Poisson algebra $(C^\infty(M), \{,\}_\theta)$, there is a natural map $C^\infty(M) \to TM : f \mapsto X_f$ between smooth functions in $C^\infty(M)$ and vector fields in TM such that

$$X_f(g(y)) \equiv \{g, f\}_\theta(y) = \left(\theta^{\mu\nu} \frac{\partial f(y)}{\partial y^\nu} \frac{\partial}{\partial y^\mu}\right) g(y), \tag{3.2.1.12}$$

for any $g \in C^\infty(M)$. This means that we can obtain a vector field $X_f = X_f^\mu \partial_\mu \in \Gamma(TM_y)$ from a smooth function $f \in C^\infty(M)$ defined at $y \in M$ where $X_f^\mu(y) = \theta^{\mu\nu}\frac{\partial f(y)}{\partial y^\nu}$. As long as θ is a Poisson structure of M, the above formula (3.2.1.12) between Hamiltonian function f and Hamiltonian vector field X_f is a Lie algebra homomorphism in the sense that

$$X_{\{f,g\}_\theta} = -[X_f, X_g], \tag{3.2.1.13}$$

where the right hand side is a Lie bracket between Hamiltonian vector fields.

From the above arguments, $U(1)$ gauge fields on a symplectic manifold $(M, B = \theta^{-1})$ can be transformed into a set of smooth functions

$$\begin{aligned} &\{D_\mu(y) \in C^\infty(M) | D_\mu(y) \equiv B_{\mu\nu} x^\nu(y) = B_{\mu\nu} y^\nu + \widehat{A}_\mu(y),\ \mu, \nu = 1, \cdots, 2n\} \\ &\qquad \text{where} \qquad x^\mu(y) \equiv y^\mu + \theta^{\mu\nu} \widehat{A}_\nu(y) \in C^\infty(M) \end{aligned} \tag{3.2.1.14}$$

After the map (3.2.1.12) is applied, we obtain Lie algebra homomorphism (3.2.1.13) between the Poisson algebra $(C^\infty(M), \{,\}_\theta)$and the Lie algebra $(\Gamma(TM), [,])$ of vector fields defined by

$$\{V_\mu = V_\mu^a \partial_a \in \Gamma(TM) | V_\mu(f)(y) \equiv \{D_\mu(y), f(y)\}_\theta,\ a = 1, \cdots, 2n\},$$

for any $f \in C^\infty(M)$. The vector fields $V_\mu = V_\mu^a(y)\frac{\partial}{\partial y^a} \in \Gamma(TM_y)$ take values in the Lie algebra of volume preserving diffeomorphisms ($\partial_a V_\mu^a = 0$). However, it can be shown that the vector fields $V_\mu \in \Gamma(TM)$ are related to the orthonormal frames (vielbeins) E_μ by $V_\mu = \lambda E_\mu$ where $\lambda^2 = \det V_\mu^a$. The metric is constructed from these vector fields:

$$ds^2 = \delta_{\mu\nu} E^\mu \otimes E^\nu = \lambda^2 \delta_{\mu\nu} V_a^\mu V_a^\nu dy^a \otimes dy^b,$$

where $E^\mu = \lambda V^\mu \in \Gamma(T^*M)$ are dual one-forms.

The electromagnetic fields in the symplectic space-time (M, B) manifest themselves only as a deformation of symplectic structure such that the resulting symplectic space-time is described by $(M, B + F)$ where $F = dA = L_X B$. This is equivalent to a deformation of frame bundle over space-time manifold M: $\partial_\mu \to E_\mu = E_\mu^a(y)\partial_a$, or, in terms of dual frames, $dy^\mu \to E^\mu = E_a^\mu(y) dy^a$.

$$ds^2 = \delta_{\mu\nu} dy^\mu \otimes dy^\nu \to ds^2 = \delta_{\mu\nu} E^\mu \otimes E^\nu.$$

We can show the emergence of gravity from the gauge fields starting with the action:

$$S_p = \frac{1}{4g_{YM}^2} \int d^{2n} y \{D_\mu(y), D_\nu(y)\}_\theta \{D^\mu(y), D^\nu(y)\}_\theta.$$

where g_{YM} is s $2n$-dimensional gauge coupling constant. Note that

$$\{D_\mu(y), D_\nu(y)\}_\theta = -B_{\mu\nu} + \partial_\mu \widehat{A}_\nu(y) - \partial_\nu \widehat{A}_\mu(y) + \{\widehat{A}_\mu(y), \widehat{A}_\nu(y)\}_\theta = -B_{\mu\nu} + \widehat{F}_{\mu\nu}(y), \tag{3.2.1.15}$$

and $\quad \{D_\mu(y), \{D_\nu(y), D_\lambda(y)\}_\theta\}_\theta = \partial_\mu \widehat{F}_{\nu\lambda}(y) + \{\widehat{A}_\mu(y), \widehat{F}_{\nu\lambda}(y)\}_\theta = \widehat{D}_\mu \widehat{F}_{\nu\lambda}(y).$ (3.2.1.16)

By identifying $f(y) = D_\mu(y)$ and $g(y) = D_\nu(y)$ with the relation of (3.2.1.15), the Lie algebra homomorphism (3.2.1.13) leads to the following identity

$$X_{\widehat{F}_{\mu\nu}} = [V_\mu, V_\nu],$$

where $V_\mu \equiv X_{D_\mu}$ and $V_\nu \equiv X_{D_\nu}$ and using (3.2.1.16) we have

$$X_{\widehat{D}_\mu \widehat{F}_{\nu\lambda}} = [V_\mu, [V_\nu, V_\lambda]].$$

Thus the equation of motion and the Jacobi identity can be written as

$$\{D^\mu, \{D_\mu, D_\nu\}_\theta\}_\theta = \widehat{D}^\mu \widehat{F}_{\mu\nu} = 0,$$
$$\{D_{[\mu}, \{D_\nu, D_{\lambda]}\}_\theta\}_\theta = \widehat{D}_{[\mu} \widehat{F}_{\nu\lambda]} = 0.$$

With the help of the above formula we have the following insightful correspondence

$$\widehat{D}_{[\mu} \widehat{F}_{\nu\lambda]} = 0 \quad \Leftrightarrow \quad [V_{[\mu}, [V_\nu, V_{\lambda]}]] = 0,$$
$$\widehat{D}^\mu \widehat{F}_{\mu\nu} = 0 \quad \Leftrightarrow \quad [V^\mu, [V_\mu, V_\nu]] = 0.$$

These relations reduce to the Einstein field equations and the first Bianchi identity for the Riemann tensor

$$[V^\mu, [V_\mu, V_\nu]] = 0 \quad \Leftrightarrow \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu},$$
$$[V_{[\mu}, [V_\nu, V_{\lambda]}]] = 0 \quad \Leftrightarrow \quad R_{[\mu\nu\lambda]\rho} = 0.$$

where the 2nd equation above implies that individual Riemann curvature components can be

$$[V_\mu, [V_\nu, V_\lambda]] = R_{\mu\nu\lambda}{}^\rho V_\rho \tag{3.2.1.17}$$

This equation will be of relevance to us later, in the next subsection as we shall see.

3.2.2 Gauge Fields from Euclidean Schwarzschild

Starting with the Euclidean Schwarzschild metric, given by:

$$ds^2 = \left(1 - \frac{2m}{r}\right) d\tau^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\Theta^2 + \sin^2\Theta \, d\phi^2), \tag{3.2.2.1}$$

we will study the symplectic gauge fields corresponding to this metric, and then will study the geometry of the vector field tetrads arising from the gauge fields, and verify if it is self dual or not. For now, our first requirement will be to construct a new co-ordinate chart that will serve our purpose.

The Darboux chart

The Darboux Theorem [85] states that we can always locally eliminate dynamical gauge fields that fluctuate about the background vacuum condensate through a local co-ordinate transformation. In general relativity, the Equivalence Principle states that there always exists a diffeomorphism that equates a curved manifold locally to a flat manifold. This theorem applies for Riemannian manifolds.

Thus, the Darboux Theorem is the equivalence principle for Symplectic manifolds. It essentially states that the symplectic structure on a curved manifold can always be equated to the symplectic structure on a flat manifold via a diffeomorphism. It can be summed up by the mathematical statement below:

$$\exists \quad \frac{\partial y^{\mu}}{\partial \xi^{a}} \quad s.t. \quad \mathcal{F}_{\mu\nu}(x)\frac{\partial y^{\mu}}{\partial \xi^{a}}\frac{\partial y^{\nu}}{\partial \xi^{b}} = B_{ab} \tag{3.2.2.2}$$

The question here is what kind of diffeomorphism will satisfy equation (3.2.2.2). The crudest answer we can give so far requires that we first write the perturbed symplectic structure as:

$$\mathcal{F}_{\mu\nu}(x) = B_{\mu\nu} + \lambda F_{\mu\nu}(x)$$

such that λ sets the strength of the dynamical field perturbation to the symplectic structure.

In the case of a given metric, we can compute the individual curvature components. Embedded within the curvature are the various $SU(2)_{\pm}$ gauge field components.

$$R_{ab} = \eta^{i(+)}_{ab} F^{i(+)} + \eta^{i(-)}_{ab} F^{i(-)} \qquad \Rightarrow \qquad F^{i(\pm)} = \frac{1}{4}\eta^{i(\pm)}_{ab} R_{ab}$$

The simplest way to eliminate local dynamical gauge fields upon switching to the Darboux co-ordinates, is to eliminate the individual $SU(2)_{\pm}$ gauge fields. This is necessarily true as we shall see below. It is known that in maximally symmetric spaces, we can have the curvature in the form:

$$R_{abcd} = g^{ij}(\vec{x})\ \varepsilon_{iab}\ \varepsilon_{jcd}$$

In the case of self-dual curvature and fields, we can further elaborate it as:

$$R_{ab} = \alpha^{(+)}_{ij}(\vec{x})\eta^{i(+)}_{ab}\eta^{j(+)}_{cd} + \alpha^{(-)}_{ij}(\vec{x})\eta^{i(-)}_{ab}\eta^{j(-)}_{cd} \quad \Rightarrow \quad F^{i(\pm)} = \frac{1}{2}\alpha^{ij(\pm)}\eta^{j(\pm)}_{ab} e^{a} \wedge e^{b}$$

where all the $\alpha^{ij(\pm)}(\vec{x})$ tensor components are diagonal (ie. $\alpha^{ij(\pm)}(\vec{x}) = 0$ for $i \neq j$). This means that the dynamical gauge field strength affiliated with the metric as a linear combination of the individual components using the t'Hooft symbols as a basis.

$$F = c^{i(+)}F^{i(+)} + c^{i(-)}F^{i(-)}$$

$$\Rightarrow \qquad F_{ab} = c^{i(+)}\alpha^{ij(+)}(\vec{x})\eta^{j(+)}_{ab} + c^{i(-)}\alpha^{ij(-)}(\vec{x})\eta^{j(-)}_{ab}$$

Now, since these t'Hooft symbols never share the same non-zero matrix elements in the same positions, we can say that the $SU(2)_{\pm}$ gauge fields are linearly independent 2-forms. From linear algebra, we know that this implies that:

$$F = 0 \quad \longleftrightarrow \quad \alpha^{ij(\pm)}(\vec{x}) = 0 \quad \Rightarrow \quad F^{i(\pm)} = 0 \quad \longleftrightarrow \quad R_{ab} = 0$$

This consequently eliminates the curvature as well, which describes the equivalence principle. Thus, if we can choose a local co-ordinate frame that locally eliminates the curvature, we will also have found the Darboux co-ordinates. We need local co-ordinates to obtain and analyze the gauge fields related to the metric. To do this, we could define a local co-ordinate system which preserves the volume element formed by the tetrads of (3.2.2.1).

$$
\begin{aligned}
\nu = \nu' &= \epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3 \wedge \epsilon^4 \\
\Rightarrow \qquad e^1 \wedge e^2 \wedge e^3 \wedge e^4 &= dt \wedge \left(r^2 dr\right) \wedge (\sin\Theta \; d\Theta) \wedge d\phi
\end{aligned}
\tag{3.2.2.3}
$$

These co-ordinates are known as the Darboux co-ordinates, the principle behind this design being to make the tetrads equivalent to the exact differentials of the local choice of co-ordinates.

$$
X^a = \{\tau, \rho, x, y\} = \left\{t, \frac{r^3}{3}, -\cos\Theta, \phi\right\} \tag{3.2.2.4}
$$

The metric, in these co-ordinates are then written as:

$$
ds^2 = \widetilde{f}(\rho) d\tau^2 + \frac{1}{\widetilde{f}(\rho)} \frac{d\rho^2}{(3\rho)^{\frac{4}{3}}} + (3\rho)^{\frac{2}{3}} \left\{ \frac{dx^2}{1-x^2} + \left(1-x^2\right) dy^2 \right\}, \qquad \widetilde{f}(\rho) = 1 - \frac{2m}{(3\rho)^{\frac{1}{3}}} \tag{3.2.2.5}
$$

Thus, for the inverse tetrads we have:

$$
\begin{aligned}
\left(\frac{\partial}{\partial s}\right)^2 &= \mathcal{E}_a \otimes \mathcal{E}_a = \lambda^{-2} V_a \otimes V_a \\
&= \widetilde{f}^{-1}(\rho) \left(\frac{\partial}{\partial \tau}\right)^2 + \widetilde{f}(\rho)(3\rho)^{\frac{4}{3}} \left(\frac{\partial}{\partial \rho}\right)^2 + \frac{1}{(3\rho)^{\frac{2}{3}}} \left\{ \left(1-x^2\right) \left(\frac{\partial}{\partial x}\right)^2 + \frac{1}{(1-x^2)} \left(\frac{\partial}{\partial y}\right)^2 \right\}
\end{aligned}
\tag{3.2.2.6}
$$

Looking at the metric (3.2.2.1) again, one can easily write the two matrices:

$$
\epsilon^a = \begin{pmatrix} f^{\frac{1}{2}}(r) & 0 & 0 & 0 \\ 0 & f^{-\frac{1}{2}}(r) & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\Theta \end{pmatrix} \qquad \mathcal{E}_a = \begin{pmatrix} f^{-\frac{1}{2}}(r) & 0 & 0 & 0 \\ 0 & f^{\frac{1}{2}}(r) & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r\sin\Theta} \end{pmatrix} \tag{3.2.2.7}
$$

Using the Darboux co-ordinates of (3.2.2.4), we can define a symplectic form:

$$
\omega = \epsilon^1 \wedge \epsilon^2 + \epsilon^3 \wedge \epsilon^4 \; = \; d\tau \wedge d\rho + dx \wedge dy \; = \; r^2 dt \wedge dr + \sin\Theta \; d\Theta \wedge d\phi \tag{3.2.2.8}
$$

such that one can re-obtain the original volume form ν

$$
\nu = \frac{1}{2}\omega \wedge \omega = r^2 \sin\Theta \; dt \wedge dr \wedge d\Theta \wedge d\phi
$$

that was shown in (3.2.2.3).

Complex Stereographic Projection - an alternate choice of coordinates

Now it is understood that the polar co-ordinate system chosen here results in a multi-valuedness towards the poles that causes a breakdown of the one-to-one correspondence between the Cartesian and polar variables, certifying a diffeomorphism, since the azimuthal angle ϕ is now arbitrary.

$$
(x, y, z) \longleftrightarrow (r, \Theta, \phi) \qquad\qquad (0, 0, \pm r) \longleftrightarrow (r, 0, ?)
$$

Thus, one needs to consider an alternate chart that preserves the correspondence. One such choice of local co-ordinates is the complex stereographic projection. There are two different charts for two different localities :

$$\mathbb{C} = U_+ = S^2 - \{x_\infty\} \quad : \quad (x, y, z) \longleftrightarrow \left(r, Z_+, \bar{Z}_+\right) \quad \text{where} \quad Z_+ = \frac{x + iy}{r - z} \tag{3.2.2.9}$$

$$\bar{\mathbb{C}} = U_- = S^2 - \{x_0\} \quad : \quad (x, y, z) \longleftrightarrow \left(r, Z_-, \bar{Z}_-\right) \quad \text{where} \quad Z_- = \frac{x - iy}{r + z} \tag{3.2.2.10}$$

where locality $\mathbb{C}$ describes the entire sphere except for the north pole, while $\bar{\mathbb{C}}$ describes the same sphere, only this time exempting the south pole, both with no arbitrary values in their localities:

$$U_- \ : \ (0, 0, \ \ r) \longleftrightarrow (r, 0, 0)$$
$$U_+ \ : \ (0, 0, -r) \longleftrightarrow (r, 0, 0)$$

The correspondence to the polar co-ordinates is given by:

$$\begin{gathered} Z_+ = \frac{e^{i\phi} \sin \Theta}{1 - \cos \Theta} = e^{i\phi} \cot \frac{\Theta}{2}, \qquad Z_- = (Z_+)^{-1} \qquad dZ_+ = -(Z_-)^{-2} \, dZ_- \\ Z_+ \bar{Z}_- = e^{2i\phi} \qquad \left(\bar{Z}_+\right)^{-1} Z_- = \tan \frac{\Theta}{2} \end{gathered} \tag{3.2.2.11}$$

However, to preserve the volume element under this diffeomorphism we need to obtain the appropriate tetrad. This can be done by adjusting the wedge product:

$$-2i \left(\frac{1 - \cos \Theta}{2}\right)^2 dZ_+ \wedge d\bar{Z}_+ = \sin \Theta \ d\Theta \wedge d\phi$$

$$|Z_+|^2 = \frac{\sin^2 \Theta}{(1 - \cos \Theta)^2} = \frac{1 + \cos \Theta}{1 - \cos \Theta} \quad \Rightarrow \quad 1 + |Z_+|^2 = \frac{2}{1 - \cos \Theta}$$

$$\therefore \quad \xi_+ = -2i \frac{dZ_+ \wedge d\bar{Z}_+}{\left(1 + |Z_+|^2\right)^2} = \sin \Theta \ d\Theta \wedge d\phi \qquad \omega = *\xi_+ + \xi_+ \tag{3.2.2.12}$$

This 2-form holds the same (form invariant) expression in the other locality as well :

$$dZ_+ \wedge d\bar{Z}_+ = |Z_-|^{-4} dZ_- \wedge d\bar{Z}_- \qquad\qquad 1 + |Z_+|^2 = |Z_-|^{-2} \left(1 + |Z_-|^2\right) \tag{3.2.2.13}$$

$$\therefore \quad \xi_- = -2i \frac{dZ_- \wedge d\bar{Z}_-}{\left(1 + |Z_-|^2\right)^2} \qquad \omega = *\xi_- + \xi_- \tag{3.2.2.14}$$

The respective volume element is given by:

$$\nu = \frac{1}{2} \omega \wedge \omega = -i \frac{r^2}{\left(1 + |Z_\pm|^2\right)^2} dt \wedge dr \wedge dZ_\pm \wedge d\bar{Z}_\pm$$

This 2-form's closure implies a potential field A, given by (3.2.2.8), (3.2.2.12) and (3.2.2.14) as:

$$d\omega_\pm = 0 \quad \Rightarrow \quad \omega_\pm = dA_\pm = d\left(-\frac{r^3}{3} dt + i \frac{Z_\pm d\bar{Z}_\pm - \bar{Z}_\pm dZ_\pm}{1 + |Z_\pm|^2}\right)$$

$$\Rightarrow \quad A_\pm = -\frac{r^3}{3} dt + i \frac{Z_\pm d\bar{Z}_\pm - \bar{Z}_\pm dZ_\pm}{1 + |Z_\pm|^2} + d\varphi$$

Naturally, there is a chance of a constant or a first order exterior derivative seperating the two potential form representations. To describe the connection between A_+ and A_- in the region $U_+ \cap U_-$, using (3.2.2.11) and (3.2.2.13) we have the following results:

$$dA_+ = dA_- \quad \Rightarrow \quad A_+ = A_- + d\varphi \tag{3.2.2.15}$$

$$Z_+ d\bar{Z}_+ - \bar{Z}_+ dZ_+ = -\frac{Z_- d\bar{Z}_- - \bar{Z}_- dZ_-}{|Z_-|^4}$$

$$A_+ + \frac{r^3}{3}dt = -\frac{1}{|Z_-|^2}\left(A_- + \frac{r^3}{3}dt\right) \quad \Rightarrow \quad A_+ - A_- = -i\frac{Z_- d\bar{Z}_- - \bar{Z}_- dZ_-}{|Z_-|^2} \tag{3.2.2.16}$$

Now, we can say that for a complex number:

$$\frac{z\ d\bar{z} - \bar{z}\ dz}{|z|^2} = -2i\ d\left(\arg(z)\right)$$

Thus, we can say that:

$$A_+ - A_- = -2\ d\left(\arg(Z_-)\right) = 2\ d\left(\arg(Z_+)\right)$$

$$\boxed{A_+ = A_- + 2\ d\left(\arg(Z_+)\right)} \tag{3.2.2.17}$$

Thus, as we can see that the two potentials for the two different localities, despite the same field strength form have a slight difference equivalent to the exterior derivative of the angular phase of the complex number. Now we proceed to obtain the symplectic gauge fields associated with the metric and study its salient properties.

Symplectic Analysis

Using the Darboux co-ordinates, we can obtain a symplectic gauge field set (recall eq.(3.2.1.14)):

$$C_a = B_{ab}X^b, \qquad \theta^{ab} = \frac{1}{2}\eta^3_{ab} \quad \Rightarrow \quad B_{ab} = -2\eta^3_{ab} \qquad \text{where} \qquad \eta^3_{ab} = \eta^{3(+)}_{ab} (\text{ see Appendix 6.1}).$$

In matrix form the set of symplectic gauge fields are

$$C = -2\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} \tau \\ \rho \\ x \\ y \end{pmatrix} = -2\begin{pmatrix} \rho \\ -\tau \\ y \\ -x \end{pmatrix} = -2\begin{pmatrix} \frac{1}{3}r^3 \\ -t \\ \phi \\ \cos\Theta \end{pmatrix}$$

$$\therefore \qquad C_1 = -\frac{2}{3}r^3, \qquad C_2 = 2t, \qquad C_3 = -2\phi \qquad C_4 = -2\cos\Theta \tag{3.2.2.18}$$

We can now derive the vector fields corresponding to the symplectic gauge fields (3.2.2.18) as the adjoint operation in the Poisson algebra and the result is shown in matrix form :

$$V_a(f) = \theta(C_a, f) \qquad V^\mu_a = -\theta^{\mu\nu}\partial_\nu C_a$$

$$\therefore \quad V = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} \partial_t \\ \partial_r \\ \partial_\Theta \\ \partial_\phi \end{pmatrix}\begin{pmatrix} \frac{1}{3}r^3 & -t & \phi & \cos\Theta \end{pmatrix} = \begin{pmatrix} r^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin\Theta \end{pmatrix} \tag{3.2.2.19}$$

We have the formula to relate the vector field with the tetrads:

$$V_a = \lambda E_a \qquad v^a = \lambda^{-1} e^a \tag{3.2.2.20}$$

To determine the value of λ, we make use of the relation:

$$\lambda^2 = \det V_a^\mu = r^2 \sin\Theta \quad \Rightarrow \quad \lambda = r\sqrt{\sin\Theta}$$

Now, the determinants of the volume preserving vector field array V_a^μ and that of the inverse vector field array, or corresponding tetrad array are given by:

$$\det(V_a^\mu) = \begin{vmatrix} r^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin\Theta \end{vmatrix} = r^2 \sin\Theta \qquad \det(V_\mu^a) = \begin{vmatrix} \frac{1}{r^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\sin\Theta} \end{vmatrix} = \frac{1}{r^2 \sin\Theta}$$

Knowing that $\lambda^2 = r^2 \sin\Theta$ we can say that:

$$\det(V_\mu^a) = \frac{1}{\lambda^2} \quad \Rightarrow \quad \lambda^2 = \frac{1}{\det(V_\mu^a)} \quad \Rightarrow \quad v(x) = 1 \tag{3.2.2.21}$$

thus concluding that the inverse tetrad fields satisfy equation (5.145) of [86].

Bianchi identity for Symplectic gauge and Vector fields

The Jacobi and Bianchi identities are well-studied in differential geometry. Both are derivatives of a basic identity defined by:

$$d^2\omega^n = 0$$

where ω^n is an n-form. Having arisen from the same source, it is clear there is a connection between the two identities.

$$\boxed{\begin{array}{c} \{C_a, \{C_b, C_c\}_\theta\}_\theta + \{C_b, \{C_c, C_a\}_\theta\}_\theta + \{C_c, \{C_a, C_b\}_\theta\}_\theta = 0 \\ \Updownarrow \\ [V_a, [V_b, V_c]] + [V_b, [V_c, V_a]] + [V_c, [V_a, V_b]] = 0 \end{array}}$$

However, the above identities are valid only in regions where the metric is well defined. They break down in the presence of singularity as evident in electrodynamics where we find the Bianchi identity being invalid in the presence of static and dynamic charge (current) distributions.

$$\begin{array}{c} A = A_i\, dx^i \quad A_i = \{\varphi, \vec{A}\} \qquad F = dA \quad F_{ij} = \partial_i A_j - \partial_j A_i \equiv \{\vec{E}, \vec{B}\} \\ \{\rho, \vec{J}\} = \{0, \vec{0}\}: \quad dF = 0 \quad \longrightarrow \quad \vec{\nabla}.\vec{E} = 0, \quad \vec{\nabla}.\vec{B} = 0 \\ \{\rho, \vec{J}\} \neq \{0, \vec{0}\}: \quad dF \neq 0 \quad \longrightarrow \quad \vec{\nabla}.\vec{E} = \rho, \quad \vec{\nabla}.\vec{B} = 0 \end{array}$$

The Schwarzschild space with Lorentzian signature has an irremovable singularity at the origin, making the Bianchi identity invalid there. However, for the Euclidean signature metric, the singularity is removable [87] under Kruskal-Szekeres co-ordinates which means that for the Euclidean Schwarzschild instanton, the Bianchi identity is valid throughout the space. Also, remembering (3.2.1.17), we can conclude that:

$$[V_a, [V_b, V_c]] = 0 \qquad \Rightarrow \qquad R_{abc}{}^d = 0$$

showing that the local results are consistent with our emergent set-up.

Seiberg Witten map and absence of self-duality

Seiberg and Witten showed [83] that there are two equivalent descriptions - commutative and non-commutative of the low energy effective theory, depending on the regularization scheme or path integral prescription for the open string ending on a D-brane.

Since these two descriptions arise from the same open string theory depending on different regularizations, and the physics being independent of the regularization scheme, Seiberg and Witten argued that they should be equivalent. Thus there must be a space-time field redefinition between ordinary and NC gauge fields, so called the Seiberg-Witten (SW) map.

The relation for the NC field strength $\widehat{F}$ is given by [80]:

$$\{C_a, C_b\}_\theta = -B_{ab} + \widehat{F}_{ab} \quad \Rightarrow \quad \widehat{F}_{ab} = B_{ab} + \{C_a, C_b\}_\theta \tag{3.2.2.22}$$

Using the C matrix from (3.2.2.18), we can write:

$$\{C_a, C_b\}_\theta = \begin{pmatrix} 0 & 2r^2 & 0 & 0 \\ -2r^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\sin\Theta \\ 0 & 0 & -2\sin\Theta & 0 \end{pmatrix}$$

$$\therefore \qquad \widehat{F} = -2 \begin{pmatrix} 0 & 1-r^2 & 0 & 0 \\ -(1-r^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\sin\Theta \\ 0 & 0 & -(1-\sin\Theta) & 0 \end{pmatrix} \tag{3.2.2.23}$$

At this point, we recapitulate the Seiberg-Witten map between the field strengths of the two descriptions - commutative and non-commutative, given by the formula:

$$\widehat{F} = (1 + F\Theta)^{-1} F \quad \Rightarrow \quad F = \widehat{F}\left(1 - \Theta\widehat{F}\right)^{-1}$$

It is easy to see that the commutative gauge field strength $F_{\mu\nu}$

$$F = -2 \begin{pmatrix} 0 & \dfrac{1-r^2}{r^2} & 0 & 0 \\ -\dfrac{1-r^2}{r^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1-\sin\Theta}{\sin\Theta} \\ 0 & 0 & -\dfrac{1-\sin\Theta}{\sin\Theta} & 0 \end{pmatrix} \tag{3.2.2.24}$$

shows no self-duality at all, noncommutative or otherwise.

The Seiberg-Witten field equation

Now we consider the equation of motion of the gauge fields (3.2.2.24). We start by looking at the action corresponding to the gauge fields:

$$S = \frac{1}{4g_{YM}} \int d^4y \; \{C_a, C_b\}_\theta^2, \tag{3.2.2.25}$$

$$\widehat{F} - B = (1 + F\theta).^{-1}\{F - B - F\} = -G^{-1}B,$$

where we have chosen to substitute

$$G = 1 + F\theta = \begin{pmatrix} r^{-2} & 0 & 0 & 0 \\ 0 & r^{-2} & 0 & 0 \\ 0 & 0 & (\sin\Theta)^{-1} & 0 \\ 0 & 0 & 0 & (\sin\Theta)^{-1} \end{pmatrix}, \qquad (3.2.2.26)$$

into (3.2.2.25), where using (3.2.2.22), we can write the action now as:

$$S = \frac{1}{4g_{YM}} \int d^4y \left(\widehat{F} - B\right)^{\mu\nu} \left(\widehat{F} - B\right)_{\mu\nu} = -\frac{1}{4g_{YM}} \int d^4x \sqrt{\mathrm{Det}(G)} \mathrm{Tr}(G^{-1}BG^{-1}B).$$

The equation of motion can be obtained by minimizing the variation of this action. Noting that $A^{[\mu\nu]} = \frac{1}{2}(A^{\mu\nu} - A^{\nu\mu})$, the commutative equation of motion is derived as:

$$\delta S = 0 \qquad \Rightarrow \qquad \delta\left[\int d^4x \sqrt{\mathrm{Det}(G)} \mathrm{Tr}(G^{-1}BG^{-1}B)\right] = 0,$$

$$\Rightarrow \qquad \int d^4x \left[\delta\left(\sqrt{\mathrm{Det}(G)}\right) \mathrm{Tr}(G^{-1}BG^{-1}B) + \sqrt{\mathrm{Det}(G)}.\delta\left\{\mathrm{Tr}(G^{-1}BG^{-1}B)\right\}\right] = 0.$$

In operator form, we write:

$$\delta\sqrt{\mathrm{Det}(G)} = \frac{1}{2}\sqrt{\mathrm{Det}(G)} G^{-1}\delta(G) = \frac{1}{2}\sqrt{\mathrm{Det}(G)}\left(G^{-1}\right)\theta\ \delta F$$

$$\begin{aligned} G^{-1}.G = \mathbb{I} \quad \Rightarrow \quad & \delta\left((G^{-1})\right).G = -G^{-1}.\delta(G) = -G^{-1}.\theta.\delta F \\ \Rightarrow \quad & \delta\left((G^{-1})\right) = -\left(\theta.G^{-1}\right).\delta F.G^{-1} \end{aligned}$$

Thus, the minimized action variation is:

$$\int d^4x \sqrt{\mathrm{Det}(G)} \left[\left(\theta.G^{-1}\right) \mathrm{Tr}(G^{-1}BG^{-1}B) + 4(G^{-1}B\left(\theta.G^{-1}\right)BG^{-1})\right]^{\mu\nu} \delta F_{\mu\nu} = 0.$$

The variation of the gauge field F and its application into the action variation are:

$$\delta F_{\mu\nu} = \delta(\partial_\mu A_\nu - \partial_\nu A_\mu) = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu,$$

$$\therefore \qquad \int d^4x \sqrt{\mathrm{Det}(G)} \left[\left(\theta.G^{-1}\right) \mathrm{Tr}(G^{-1}BG^{-1}B) - 4(G^{-1}B\left(\theta.G^{-1}\right)BG^{-1})\right]^{[\mu\nu]} \partial_\mu \delta A_\nu = 0.$$

Thus, the resulting equation of motion that is obtained for the first time here reads as:

$$\partial_\mu\left[\sqrt{G}\left\{(\theta G^{-1})^{\mu\nu}\mathrm{Tr}(G^{-1}BG^{-1}B) - 4(\theta G^{-1}BG^{-1}BG^{-1})^{[\mu\nu]}\right\}\right] = 0. \qquad (3.2.2.27)$$

Substituting G from (3.2.2.26) into (3.2.2.27) above should give us the SW field equation for the Euclidean Schwarzschild metric which is a typical example of AF gravitational instanton.

3.2.3 Geometric Analysis

Now we proceed to analyze the various geometric and topological properties of the Euclidean Schwarzschild metric. This will involve obtaining the various topological invariants related to the metric. We will start by obtaining the curvature components of the metric.

Curvature analysis

We can extract the complete set of tetrads for the metric (3.2.2.1) as:

$$\begin{aligned} e^1 &= \sqrt{1-\frac{2m}{r}}dt & e^2 &= \frac{1}{\sqrt{1-\frac{2m}{r}}}dr \\ e^3 &= r\ d\Theta & e^4 &= r\sin\Theta\ d\varphi \end{aligned} \tag{3.2.3.1}$$

Starting with (3.2.3.1) and using Cartan's 1st torsion-free structure equation, we have:

$$\begin{aligned} {\omega^1}_2 &= -{\omega^2}_1 = \frac{m}{r^2}dt & {\omega^4}_3 &= -{\omega^3}_4 = \cos\Theta\ d\varphi \\ {\omega^3}_2 &= -{\omega^2}_3 = \sqrt{1-\frac{2m}{r}}d\Theta & {\omega^4}_2 &= -{\omega^2}_4 = \sqrt{1-\frac{2m}{r}}\sin\Theta\ d\varphi \end{aligned}$$

The overall ω (spin-connection) matrix is given by:

$${\omega^i}_j = \frac{1}{r}\begin{pmatrix} 0 & a\ e^1 & 0 & 0 \\ -a\ e^1 & 0 & -b\ e^3 & -b\ e^4 \\ 0 & b\ e^3 & 0 & -c\ e^4 \\ 0 & b\ e^4 & c\ e^4 & 0 \end{pmatrix} \quad \text{where} \quad \begin{cases} a = \frac{m}{r\sqrt{f(r)}} \\ b = \sqrt{f(r)} \\ c = \cot\theta \end{cases} \tag{3.2.3.2}$$

For the curvature components, we use the 2nd structure equation:

$${R^i}_j = d{\omega^i}_j + {\omega^i}_k \wedge {\omega^k}_j \tag{3.2.3.3}$$

Combining (3.2.3.2) and (3.2.3.3) gives the following non-vanishing curvature components:

$$\begin{aligned} {R^1}_{212} &= -{R^1}_{221} = -{R^2}_{112} = {R^2}_{121} = \frac{2m}{r^3} \\ {R^1}_{313} &= -{R^1}_{331} = -{R^3}_{113} = {R^3}_{131} = -\frac{m}{r^3} \\ {R^1}_{414} &= -{R^1}_{441} = -{R^4}_{114} = {R^4}_{141} = -\frac{m}{r^3} \\ {R^2}_{323} &= -{R^2}_{332} = -{R^3}_{223} = {R^3}_{232} = -\frac{m}{r^3} \\ {R^2}_{424} &= -{R^2}_{442} = -{R^4}_{224} = {R^4}_{242} = -\frac{m}{r^3} \\ {R^3}_{434} &= -{R^3}_{443} = -{R^4}_{334} = {R^4}_{343} = \frac{2m}{r^3} \end{aligned} \tag{3.2.3.4}$$

In a compact form, the R_{ab} matrix can be written as:

$$R_{ab} = \frac{m}{r^3}\begin{pmatrix} 0 & 2x & -y & -z \\ -2x & 0 & -\bar{z} & \bar{y} \\ y & \bar{z} & 0 & 2\bar{x} \\ z & -\bar{y} & -2\bar{x} & 0 \end{pmatrix} \tag{3.2.3.5}$$

where we use the representation:

$$\begin{aligned} x &= e^1\wedge e^2 & y &= e^1\wedge e^3 & z &= e^1\wedge e^4 \\ \bar{x} &= e^3\wedge e^4 & \bar{y} &= e^4\wedge e^2 & \bar{z} &= e^2\wedge e^3 \end{aligned} \qquad \text{and} \qquad x\wedge\bar{x} = y\wedge\bar{y} = z\wedge\bar{z} = \nu.$$

where ν is the volume form. Clearly, we can see that R_{ab} matrix of (3.2.3.5) is not self dual since each of its components are made of only one 2-form term, making it impossible to exhibit self-duality.

$$*R_{ab} = \frac{1}{2}\frac{\varepsilon_{ab}{}^{cd}}{\sqrt{g}}R_{cd} \neq R_{ab}$$

With the Riemann tensor components (3.2.3.4), we can compute the Ricci tensor and scalar

$$R_{ij} = \eta^{kl}R_{ikjl} = \eta_{im}\eta^{kl}R^{m}{}_{kjl} \qquad \Rightarrow \qquad R_{11} = R_{22} = R_{33} = R_{44} = 0$$
$$R = \eta^{ij}R_{ij} \qquad \Rightarrow \qquad R = 0 \tag{3.2.3.6}$$

So the Euclidean Schwarzschild solution classified in the literature as AF gravitational instanton does not exhibit self-duality although it is a Ricci-flat manifold. Since the spin connections in eq. (3.2.3.2) are neither self-dual or anti-self dual, we can proceed to construct both type of SU(2) gauge fields and the field strengths using respectively the spin connections (3.2.3.2) and curvature components (3.2.3.5) using the following formula:

$$A^{(\pm)i} = \frac{1}{4}\eta^{(\pm)i}_{\mu\nu}\omega_{\mu\nu} \qquad\qquad F^{(\pm)i} = \frac{1}{4}\eta^{(\pm)i}_{\mu\nu}R_{\mu\nu} \tag{3.2.3.7}$$

where the symbols $\eta^{(\pm)i}_{\mu\nu}$ are the t'Hooft symbols (see Appendix Sec. 6.1). By construction the field strengths should be either self-dual (for the + sign) or anti-self dual (for the - sign). According to a general result (3.41) found in [78], the SU(2) gauge field (3.2.3.7) automatically satisfy the self duality equation and hence these solution describes an SU(2) Yang-Mills (anti) instanton on the space (3.2.2.1).

Thus, we have the following description for the $SU(2)_+$ instanton and $SU(2)_-$ anti-instanton gauge fields respectively listed as :

$$A^{(+)1} = -\frac{1}{4r}Tr\begin{pmatrix} 0 & b\,e^4 & c\,e^4 & 0 \\ 0 & b\,e^3 & 0 & -c\,e^4 \\ a\,e^1 & 0 & b\,e^3 & b\,e^4 \\ 0 & -a\,e^1 & 0 & 0 \end{pmatrix} = -\frac{1}{2r}b\,e^3 \qquad A^{(-)1} = -\frac{1}{4r}Tr\begin{pmatrix} 0 & -b\,e^4 & -c\,e^4 & 0 \\ 0 & b\,e^3 & 0 & -c\,e^4 \\ a\,e^1 & 0 & b\,e^3 & b\,e^4 \\ 0 & a\,e^1 & 0 & 0 \end{pmatrix} = -\frac{1}{2r}b\,e^3$$

$$A^{(+)2} = -\frac{1}{4r}Tr\begin{pmatrix} 0 & -b\,e^3 & 0 & c\,e^4 \\ 0 & b\,e^4 & c\,e^4 & 0 \\ 0 & a\,e^1 & 0 & 0 \\ a\,e^1 & 0 & b\,e^3 & b\,e^4 \end{pmatrix} = -\frac{1}{2r}b\,e^4 \qquad A^{(-)2} = -\frac{1}{4r}Tr\begin{pmatrix} 0 & -b\,e^3 & 0 & c\,e^4 \\ 0 & -b\,e^4 & -c\,e^4 & 0 \\ 0 & a\,e^1 & 0 & 0 \\ -a\,e^1 & 0 & -b\,e^3 & -b\,e^4 \end{pmatrix} = -\frac{1}{2r}b\,e^4$$

$$A^{(+)3} = -\frac{1}{4r}Tr\begin{pmatrix} -a\,e^1 & 0 & -b\,e^3 & -b\,e^4 \\ 0 & -a\,e^1 & 0 & 0 \\ 0 & b\,e^4 & c\,e^4 & 0 \\ 0 & -b\,e^3 & 0 & c\,e^4 \end{pmatrix} = \frac{1}{2r}\left(a\,e^1 - c\,e^4\right) \qquad A^{(-)3} = -\frac{1}{4r}Tr\begin{pmatrix} -a\,e^1 & 0 & -b\,e^3 & -b\,e^4 \\ 0 & -a\,e^1 & 0 & 0 \\ 0 & -b\,e^4 & -c\,e^4 & 0 \\ 0 & b\,e^3 & 0 & -c\,e^4 \end{pmatrix} = \frac{1}{2r}\left(a\,e^1 + c\,e^4\right) \tag{3.2.3.8}$$

$$
\begin{aligned}
F^{(+)1} &= -\frac{m}{4r^3} Tr \begin{pmatrix} z & -\bar{y} & -2\bar{x} & 0 \\ y & \bar{z} & 0 & 2\bar{x} \\ 2x & 0 & \bar{z} & -\bar{y} \\ 0 & -2x & y & z \end{pmatrix} & F^{(-)1} &= -\frac{m}{4r^3} Tr \begin{pmatrix} -z & \bar{y} & 2\bar{x} & 0 \\ y & \bar{z} & 0 & 2\bar{x} \\ 2x & 0 & \bar{z} & \bar{y} \\ 0 & 2x & -y & -z \end{pmatrix} \\
&= -\frac{m}{2r^3}(z+\bar{z}) & &= \frac{m}{2r^3}(z-\bar{z}) \\
F^{(+)2} &= -\frac{m}{4r^3} Tr \begin{pmatrix} -y & -\bar{z} & 0 & -2\bar{x} \\ z & -\bar{y} & -2\bar{x} & 0 \\ 0 & 2x & -y & -z \\ 2x & 0 & \bar{z} & -\bar{y} \end{pmatrix} & F^{(-)2} &= -\frac{m}{4r^3} Tr \begin{pmatrix} -y & -\bar{z} & 0 & -2\bar{x} \\ -z & \bar{y} & 2\bar{x} & 0 \\ 0 & 2x & -y & -z \\ -2x & 0 & -\bar{z} & \bar{y} \end{pmatrix} \\
&= \frac{m}{2r^3}(y+\bar{y}) & &= \frac{m}{2r^3}(y-\bar{y}) \\
F^{(+)3} &= -\frac{m}{4r^3} Tr \begin{pmatrix} -2x & 0 & -\bar{z} & \bar{y} \\ 0 & -2x & y & z \\ z & -\bar{y} & -2\bar{x} & 0 \\ -y & -\bar{z} & 0 & -2\bar{x} \end{pmatrix} & F^{(-)3} &= -\frac{m}{4r^3} Tr \begin{pmatrix} -2x & 0 & -\bar{z} & -\bar{y} \\ 0 & -2x & y & z \\ -z & \bar{y} & 2\bar{x} & 0 \\ y & \bar{z} & 0 & 2\bar{x} \end{pmatrix} \\
&= \frac{m}{r^3}(x+\bar{x}) & &= \frac{m}{r^3}(x-\bar{x})
\end{aligned} \tag{3.2.3.9}
$$

Remembering that the curvature components are given by (3.2.3.3), we can write:

$$R^a{}_b = \frac{1}{2} R^a{}_{b\mu\nu}\, dx^\mu \wedge dx^\nu \quad \Rightarrow \quad R^a{}_{bcd} = \iota_{E_d}\iota_{E_c} R^a{}_b$$

$$\iota_{E_d}\iota_{E_c}\left(d\omega^a{}_b + \omega^a{}_m \wedge \omega^m{}_b\right) = \{\partial_c(\omega_d{}^a{}_b) - \omega_c{}^m{}_d\ \omega_m{}^a{}_b + \omega_c{}^a{}_m\ \omega_d{}^m{}_b\}$$

$$\therefore \quad R^a{}_{bcd} = \{\nabla_c(\omega_d{}^a{}_b) - \omega_c{}^m{}_d\ \omega_m{}^a{}_b + \omega_c{}^a{}_m\ \omega_d{}^m{}_b\}$$

Thus we get the Ricci tensor to be

$$R_{ac} = \nabla_c(f_{bab}) - \omega_{cmb}\,\omega_{mab} - \omega_{cma}\ f_{bmb}, \qquad \nabla_c(f_{bab}) = 0.$$

Finally the Ricci scalar can be written as

$$R = -\{\omega_{amb}\,\omega_{mab} + f_{ama}\ f_{bmb}\}$$

Now, since the Ricci scalar vanishes in our case (see eqn. (3.2.3.6)), we have:

$$R = 0 \quad \Rightarrow \quad (f_{aba})^2 = \omega_{abc}\,\omega_{cab} \tag{3.2.3.10}$$

From the Cartan structure equations, $T^a = de^a + \omega^a{}_b \wedge e^b$, under torsion free condition ($T^a = 0$), we have:

$$de^a = -\omega^a{}_b \wedge e^b \quad \Rightarrow \quad \partial_\mu e^a{}_\nu = -\omega_\mu{}^a{}_b e^b{}_\nu$$

Upon contraction with $E_c{}^\nu$ (where $E_a{}^\nu = (e^a{}_\nu)^{-1}$), we can proceed to write:

$$E_c{}^\nu \partial_\mu e^a{}_\nu = -\omega_\mu{}^a{}_b\big(e^b{}_\nu E_c{}^\nu\big) \qquad \Rightarrow \qquad \underbrace{\partial_\mu\big(E_c{}^\nu e^a{}_\nu\big)}_{0} - e^a{}_\nu \partial_\mu E_c{}^\nu = -\omega_\mu{}^a{}_b \delta^b_c = -\omega_\mu{}^a{}_c$$

$$\therefore \quad \partial_\mu E_c{}^\nu = \omega_\mu{}^a{}_c E_a{}^\nu \tag{3.2.3.11}$$

The tetrads and the vector fields in (3.2.2.19)-(3.2.2.20) exhibit the structure equations:

$$[E_a, E_b] = -f_{ab}{}^c E_c \qquad [V_a, V_b] = -g_{ab}{}^c V_c \tag{3.2.3.12}$$

If the vector fields $\{E_a\}$ and $\{V_a\}$ are related by (3.2.2.20), then we can suppose that:

$$dV_b = d(\lambda E_b) = d\lambda \wedge E_b - \lambda \omega^c{}_b E_c = d(\log\lambda) \wedge V_b - \omega^c{}_b V_c$$

$$\iota_{V_a} dV_b = V_a V_b = V_a(\log\lambda) V_b - \lambda\, \omega_a{}^c{}_b V_c$$

$$\therefore \quad [V_a, V_b] = V_a(\log\lambda) V_b - V_b(\log\lambda) V_a - \lambda\left(\omega_a{}^c{}_b - \omega_b{}^c{}_a\right) V_c$$

$$\Rightarrow \quad -g_{ab}{}^c V_c = \frac{1}{2}\left(g_{ma}{}^m V_b - g_{mb}{}^m V_a\right) - \lambda\, f_{ab}{}^c V_c$$

So we can write the structure constants in terms of the metric

$$f_{ab}{}^c = \frac{1}{\lambda}\left\{g_{ab}{}^c + \frac{1}{2}\left(g_{ma}{}^m \delta_b^c - g_{mb}{}^m \delta_a^c\right)\right\} \quad \Rightarrow \quad f_{ab}{}^a = \frac{1}{\lambda} g_{ab}{}^a \tag{3.2.3.13}$$

We also note the relation between spin connection and structure constant:

$$\omega_{abc} = \frac{1}{2}\Big(f_{abc} - f_{bca} + f_{cab}\Big) \tag{3.2.3.14}$$

Finally, an important identity here is:

$$\rho^b = g_a{}^{ba} \qquad \Psi^d = \frac{1}{2}\varepsilon^{abcd} g_{abc} \tag{3.2.3.15}$$

$$\rho_b \rho^b = \Psi_d \Psi^d \quad \Rightarrow \quad \rho^a = \pm \Psi^a \tag{3.2.3.16}$$

With a little effort, it can be shown (in any $2n$-dimensions) [74, 84] that the right-hand side of the Bianchi identity for vector fields is precisely equivalent to the first Bianchi identity of Riemann curvature tensors, i.e.,

$$[V_a, [V_b, V_c]] + \text{cyclic} = 0 \qquad \Leftrightarrow \qquad R_{[abc]d} = 0, \tag{3.2.3.17}$$

where $[abc]$ denotes the cyclic permutation of indices. The equation (3.2.3.17) leads to a cryptic result for Ricci tensors [74, 84]

$$R_{ab} = -\frac{1}{\lambda^2}\left[g_d^{(+)i} g_d^{(-)j}\left(\eta^i_{ac}\bar{\eta}^j_{bc} + \eta^i_{bc}\bar{\eta}^j_{ac}\right) - g_c^{(+)i} g_d^{(-)j}\left(\eta^i_{ac}\bar{\eta}^j_{bd} + \eta^i_{bc}\bar{\eta}^j_{ad}\right)\right] \tag{3.2.3.18}$$

where η^i_{ab} and $\bar{\eta}^i_{ab}$ are self-dual and anti-self-dual t'Hooft symbols. To get the result (3.2.3.18), we have to define the canonical decomposition of the structure equation (3.2.3.12) like

$$g_{abc} = g_c^{(+)i} \eta^i_{ab} + g_c^{(-)i} \bar{\eta}^i_{ab}. \tag{3.2.3.19}$$

A notable point is that the right-hand side of (3.2.3.18) consists of purely interaction terms between self-dual and anti-self-dual parts in (3.2.3.19) which is the feature withheld by matter fields only [79]. A gravitational instanton which is a Ricci-flat, Kähler manifold can be understood as either $g_c^{(-)i} = 0$ (self-dual) or $g_c^{(+)i} = 0$ (anti-self-dual) in terms of (3.2.3.19) and so $R_{ab} = 0$ in (3.2.3.18). Hence, the result (3.2.3.18) is consistent with the Ricci-flatness of gravitational instantons. However (3.2.3.18) also has a nontrivial trace contribution, i.e., a nonzero Ricci scalar, due to the second part which does not exist in Einstein gravity.

The content of the energy-momentum tensor defined by the right-hand side of the Bianchi identity for vector fields becomes manifest by decomposing it into two parts, denoted by $8\pi G T^{(M)}_{ab}$ and $8\pi G T^{(L)}_{ab}$, respectively [74, 84]:

$$8\pi G T^{(M)}_{ab} = -\frac{1}{\lambda^2}\Big(g_{acd}g_{bcd} - \frac{1}{4}\delta_{ab}g_{cde}g_{cde}\Big), \tag{3.2.3.20}$$

$$8\pi G T^{(L)}_{ab} = \frac{1}{2\lambda^2}\Big(\rho_a\rho_b - \Psi_a\Psi_b - \frac{1}{2}\delta_{ab}\left(\rho_c^2 - \Psi_c^2\right)\Big), \tag{3.2.3.21}$$

$$\text{where} \qquad \rho_a \equiv g_{bab}, \qquad \Psi_a \equiv -\frac{1}{2}\varepsilon^{abcd}g_{bcd}. \tag{3.2.3.22}$$

The first energy-momentum tensor (3.2.3.20) is traceless, i.e. $8\pi G T^{(M)}_{aa} = 0$, which is a consequence of the identity $\eta^i_{ab}\overline{\eta}^j_{ab} = 0$ when applied to the first part of (3.2.3.18). The Ricci scalar $R \equiv R_{aa}$ can be calculated by (3.2.3.21) to yield

$$R = \frac{1}{2\lambda^2}\Big(\rho_a^2 - \Psi_a^2\Big). \tag{3.2.3.23}$$

The equation (3.2.3.23) immediately leads to the conclusion that a four-manifold emergent from pure symplectic gauge fields (without source terms) can have a vanishing Ricci scalar if and only if (see eqn. (3.2.3.15) and (3.2.3.16) and its derivation)

$$\rho_a = \pm\Psi_a \tag{3.2.3.24}$$

that is similar to the self-duality equation. When the relation (3.2.3.24) is obeyed, the second energy- momentum tensor $8\pi G T^{(L)}_{ab}$ (3.2.3.21) identically vanishes which confirms that the space of a Euclidean Schwarzschild solution is complete vacuum with no matter present.

Topological Invariants

In gravity topology can play a role at various levels. At the macroscopic level one may consider multiplying corrected universes and wormholes, whilst at the microscopic Planck scale space-time topology may subject to quantum fluctuations; in analogy with others QFTs like sigma models and Yang-Mills theories, it is expected that the quantum tunneling process between different topologies are dominated by finite-action solutions of Euclidean gravity, the gravitational instantons.

One way to characterize topologically non-trivial solutions of the gravitational field equations is by the value of topologically invariant integral over certain polynomials of the curvature tensor. In four dimensions there are essentially two independent topological invariants the Euler Charcteristics and the Hirzebruch signature [62]. Every manifold with an associated metric has topological invariants that characterize it, implying geometric similarities between manifolds sharing the same invariant. Here, we will calculate two topological invariants of the Euclidean Schwarzschild instanton.

Euler characteristic

We can use the Riemann tensor components to compute the Euler characteristic $\chi(M)$:

$$\chi(M) = \frac{1}{32\pi^2}\int_M \varepsilon^{abcd}R_{ab}\wedge R_{cd} + \frac{1}{16\pi^2}\int_{\partial M}\varepsilon^{abcd}\left(\theta_{ab}\wedge R_{cd} - \frac{2}{3}\theta_{ab}\wedge\theta_{cp}\wedge\theta_{pd}\right) \tag{3.2.3.25}$$

where θ_{AB} is the second fundamental form of the boundary ∂M. It is defined by

$$\theta_{AB} = \omega_{AB} - \omega_{0AB}, \tag{3.2.3.26}$$

where ω_{AB} are the actual connection 1-forms and ω_{0AB} are the connection 1-forms if the metric were locally a product form near the boundary [88]. The connection 1-form ω_{0AB} will have only tangential components on ∂M and so the second fundamental form θ_{AB} will have only normal components on ∂M. The bulk part of the Euler characteristic is given by:

$$\chi_{bulk} = \frac{1}{32\pi^2}\int_M \varepsilon^{abcd} R_{ab} \wedge R_{cd} \tag{3.2.3.27}$$

To compute the expression in (3.2.3.27), we only need to consider 6 combinations, where one half is equivalent to the other half. These combinations are given as:

$$\begin{aligned} R_{12} \wedge R_{34} &= R_{34} \wedge R_{12} \\ R_{13} \wedge R_{24} &= R_{24} \wedge R_{13} \\ R_{14} \wedge R_{23} &= R_{23} \wedge R_{14} \end{aligned} \tag{3.2.3.28}$$

Since each permutation of 2 index pairs yields 2 combinations, and as shown in (3.2.3.28), equivalent pairs of combinations exist, we can say that (3.2.3.27) reduces to:

$$\chi_{bulk} = \frac{1}{4\pi^2}\int_M \left(\varepsilon^{1234} R_{12} \wedge R_{34} + \varepsilon^{1324} R_{13} \wedge R_{24} + \varepsilon^{1423} R_{14} \wedge R_{23}\right) \tag{3.2.3.29}$$

We can use the Bianchi identity for curvature tensor to show that:

$$\begin{aligned} R_{ab} \wedge R_{cd} &= d\omega_{ab} \wedge R_{cd} + \omega_{ap} \wedge \omega^{p}{}_{b} \wedge R_{cd} \\ d\omega_{ab} \wedge R_{cd} &= d\left(\omega_{ab} \wedge R_{cd}\right) \\ \omega_{am} \wedge \omega^{m}{}_{b} \wedge R_{cd} &= \omega_{ap} \wedge \omega^{p}{}_{b} \wedge d\omega_{cd} + \omega_{ap} \wedge \omega^{p}{}_{b} \wedge \omega_{cq} \wedge \omega^{q}{}_{d} \\ \therefore \qquad \omega_{ap} \wedge \omega^{p}{}_{b} \wedge R_{cd} &= d\left(\omega_{ap} \wedge \omega^{p}{}_{b} \wedge \omega_{cd}\right) + \omega_{ap} \wedge \omega^{p}{}_{b} \wedge \omega_{cq} \wedge \omega^{q}{}_{d} \end{aligned} \tag{3.2.3.30}$$

$$\begin{aligned} \int_M R_{ab} \wedge R_{cd} &= \int_M d\left(\omega_{ab} \wedge R_{cd} + \omega_{am} \wedge \omega^{m}{}_{b} \wedge \omega_{cd}\right) + \int_M \omega_{am} \wedge \omega^{m}{}_{b} \wedge \omega_{cn} \wedge \omega^{n}{}_{d}, \\ &= \int_{\partial M} \left(\omega_{ab} \wedge R_{cd} + \omega_{am} \wedge \omega^{m}{}_{b} \wedge \omega_{cd}\right) + \int_M \omega_{am} \wedge \omega^{m}{}_{b} \wedge \omega_{cn} \wedge \omega^{n}{}_{d} \end{aligned} \tag{3.2.3.31}$$

We can see that for the 2nd term in (3.2.3.30) and for the 3rd term in (3.2.3.31) that:

$$\begin{aligned} \varepsilon^{abcd}\omega_{ap} \wedge \omega^{p}{}_{b} &= \varepsilon^{abcd}\left(\omega_{ac} \wedge \omega^{c}{}_{b} + \omega_{ad} \wedge \omega^{d}{}_{b}\right) \\ \therefore \quad \varepsilon^{abcd}\omega_{ap} \wedge \omega^{p}{}_{b} \wedge \omega_{cq} \wedge \omega^{q}{}_{d} &= 0 \end{aligned} \tag{3.2.3.32}$$

Using (3.2.3.32) we can see that (3.2.3.31) becomes:

$$\int_M R_{ab} \wedge R_{cd} = \int_{\partial M} \left(\omega_{ab} \wedge R_{cd} + \omega_{am} \wedge \omega^{m}{}_{b} \wedge \omega_{cd}\right) \tag{3.2.3.33}$$

For the 2nd term, we refer to (3.2.3.2) to point out that besides the 2nd row and column, all other rows and columns have only 2 non-zero elements (the first one has only one). ie.:

$$\sum_m \varepsilon^{abcd}\omega_{ap} \wedge \omega^{p}{}_{b} \wedge \omega_{cd} = \varepsilon^{abcd}\left(\omega_{ac} \wedge \omega^{c}{}_{b} \wedge \omega_{cd} + \omega_{ad} \wedge \omega^{d}{}_{b} \wedge \omega_{cd}\right) = 0; \quad \forall \ \ c, d \neq 2$$

Thus, the different non-vanishing components of (3.2.3.33) are:

$$\int_M R_{12} \wedge R_{34} = \int_{\partial M} \omega_{12} \wedge R_{34} = \int_{\partial M} \frac{2m^2}{r^3} dt \wedge d\Theta \wedge \sin\Theta d\phi \tag{3.2.3.34}$$

$$\int_M R_{13} \wedge R_{24} = -\int_{\partial M} \omega_{12} \wedge \omega^2{}_3 \wedge \omega_{24} = -\int_{\partial M} \frac{m}{r^2}\left(1 - \frac{2m}{r}\right) dt \wedge d\Theta \wedge \sin\Theta d\phi \xrightarrow{r=2m} 0 \tag{3.2.3.35}$$

$$\int_M R_{14} \wedge R_{23} = \int_{\partial M} \omega_{12} \wedge \omega^2{}_4 \wedge \omega_{23} = -\int_{\partial M} \frac{m}{r^2}\left(1 - \frac{2m}{r}\right) dt \wedge d\Theta \wedge \sin\Theta d\phi \xrightarrow{r=2m} 0 \tag{3.2.3.36}$$

Applying (3.2.3.33), (3.2.3.34), (3.2.3.35) and (3.2.3.36) to (3.2.3.29) gives us:

$$\chi_{bulk} = \frac{1}{4\pi^2} \int_{\partial M} \omega_{12} \wedge R_{34} = \frac{1}{4\pi^2} \frac{2m^2}{r_h^3} \int_0^\beta dt \wedge \int_0^\pi \sin\Theta \; d\Theta \wedge \int_0^{2\pi} d\phi = \frac{2m^2}{\pi r_h^3}\beta \tag{3.2.3.37}$$

Here, we compactify the imaginary time, such that it lies within the range: $0 \leq t \leq \beta$ (generalization of the condition of the removal of conical singularity for our class of metrics). The upper limit β (realized as inverse temperature for the black hole) is given by:

$$\kappa\beta = 2\pi \qquad \text{where} \quad \kappa = \frac{1}{2} \frac{\partial_r g_{tt}}{\sqrt{g_{tt} g_{rr}}}\bigg|_{r=r_h} = \frac{1}{2} \left(\partial_r f(r)\right)_{r=r_h} = \frac{m}{r_h^2}$$

$$\therefore \quad \beta = \frac{2\pi}{\kappa} = \frac{2\pi r_h^2}{m} \tag{3.2.3.38}$$

Thus, for Schwarzschild, $r_h = 2m$ and applying (3.2.3.38) in (3.2.3.37) the bulk part is:

$$\chi_{bulk} = \frac{4m}{r_h} = 2$$

The boundary integral term of the Euler characteristics is given by:

$$\chi_{boundary} = \frac{1}{16\pi^2} \int_{\partial M} \varepsilon^{abcd} \left(\theta_{ab} \wedge R_{cd} - \frac{2}{3} \theta_{ab} \wedge \theta_{cp} \wedge \theta_{pd}\right)$$

Recall that, the 1-form θ_{ab} is given by:

$$\theta_{ab} = \omega_{ab} - \omega_{0ab}, \qquad \text{where} \qquad \omega_{0ab} = (\omega_{ab})_{r=\infty}$$

Only the component along the normal to the surface is to be treated differently ie.:

$$\theta_{12} = \omega_{12}$$

The $\theta^a{}_b$ matrix is given by:

$$\theta^a{}_b = \begin{pmatrix} 0 & \frac{m}{r^2} dt & 0 & 0 \\ -\frac{m}{r^2} dt & 0 & \left(1 - \sqrt{1 - \frac{2m}{r}}\right) d\Theta & \left(1 - \sqrt{1 - \frac{2m}{r}}\right) \sin\Theta \; d\varphi \\ 0 & -\left(1 - \sqrt{1 - \frac{2m}{r}}\right) d\Theta & 0 & 0 \\ 0 & -\left(1 - \sqrt{1 - \frac{2m}{r}}\right) \sin\Theta \; d\varphi & 0 & 0 \end{pmatrix}$$

In this case, since $\partial M \quad \Rightarrow \quad r = \infty$, when θ_{12} vanishes as $r \to \infty$. Thus, we can effectively say, $\theta_{ab} = 0$ which corresponds to setting $\chi_{boundary} = 0$ so that we can write:

$$\chi(M) = \chi_{bulk} + \chi_{boundary} = 2 + 0 = 2 \tag{3.2.3.39}$$

which is the value of Euler characteristic for Euclidean Schwarzschild metric. (see also [89] for a similar computation which was reported there for the first time.)

Recalling how the Schwarzschild metric is a sum of an $SU(2)_L$ instanton and $SU(2)_R$ anti-instanton resulting from the $SU(2)_+$ and $SU(2)_-$ gauge fields (3.2.3.9), we can further calculate the Euler characteristics using:

$$\eta^{(\pm)i}_{\mu\nu}\eta^{(\pm)i}_{\lambda\gamma} = \delta_{\mu\lambda}\delta_{\nu\gamma} - \delta_{\mu\gamma}\delta_{\nu\lambda} \pm \varepsilon_{\mu\nu\lambda\gamma} \qquad \Rightarrow \qquad \varepsilon_{\mu\nu\lambda\gamma} = \frac{1}{2}\Big(\eta^{(+)i}_{\mu\nu}\eta^{(+)i}_{\lambda\gamma} - \eta^{(-)i}_{\mu\nu}\eta^{(-)i}_{\lambda\gamma}\Big)$$

Thus (3.2.3.27) reduces to

$$\frac{1}{32\pi^2}\int_M \varepsilon^{abcd} R_{ab} \wedge R_{cd} = \frac{1}{4\pi^2}\int_M \Big(F^{(+)i} \wedge F^{(+)i} - F^{(-)i} \wedge F^{(-)i}\Big)$$

It is straightforward to express the topological invariant in terms of $SU(2)$ gauge fields.

$$\therefore \quad \chi_{bulk} = \frac{1}{4\pi^2}\int_M \Big(F^{(+)i} \wedge F^{(+)i} - F^{(-)i} \wedge F^{(-)i}\Big) \tag{3.2.3.40}$$

We could now follow the same process as before invoking Stoke's theorem and convert (3.2.3.40) into a boundary integral using (3.2.3.33) to obtain:

$$\begin{aligned}\frac{\varepsilon^{abcd}}{32\pi^2}\int_M R_{ab} \wedge R_{cd} &= \frac{\varepsilon^{abcd}}{32\pi^2}\int_{\partial M} (\omega_{ab} \wedge R_{cd} + \omega_{am} \wedge \omega^m{}_b \wedge \omega_{cd}) \\ &= \frac{1}{4\pi^2}\int_{\partial M}\Big(A^{(+)i} \wedge F^{(+)i} - A^{(-)i} \wedge F^{(-)i}\Big) + \frac{\varepsilon^{abcd}}{32\pi^2}\int_{\partial M} \omega_{am} \wedge \omega^m{}_b \wedge \omega_{cd}\end{aligned}$$

Seeing how the 2nd integrand vanishes for most combinations, and otherwise vanishes on the boundary itself, we can focus on the 1st integrand alone.

$$\chi_{bulk} = \frac{1}{4\pi^2}\int_{\partial M}\Big(A^{(+)i} \wedge F^{(+)i} - A^{(-)i} \wedge F^{(-)i}\Big) = \chi^+_{bulk} + \chi^-_{bulk} \tag{3.2.3.41}$$

Thus, for (3.2.3.41) we can compute the Euler character bulk values using (3.2.3.8) and (3.2.3.9) as:

$$\begin{aligned}\chi^+_{bulk} &= \frac{m^2}{2r_h^3\pi}\beta + \frac{m}{4r_h^2\pi}\beta = \left(\frac{1}{16m\pi} + \frac{1}{16m\pi}\right)\beta = 1 \\ \chi^-_{bulk} &= -\frac{1}{4\pi^2}\int_{\partial M} -\frac{m}{4r^4}\left(be^3 \wedge z - be^4 \wedge y + 2ae^1 \wedge \bar{x} - 2ce^4 \wedge x\right) = 1\end{aligned}$$

For verification, we evaluate the contributions according to (3.2.3.40) using (3.2.3.9) to get:

$$\begin{aligned}\chi^+_{bulk} &= \frac{1}{4\pi^2}\int_M F^{(+)i} \wedge F^{(+)i} = \frac{m^2}{r_h^3\pi}\beta = \frac{2m}{r_h} = 1 \\ \chi^-_{bulk} &= -\frac{1}{4\pi^2}\int_M F^{(-)i} \wedge F^{(-)i} = -\frac{1}{4\pi^2}\int_M -\left(\frac{m^2}{2r^6} + \frac{m^2}{2r^6} + \frac{2m^2}{r^6}\right)\nu = 1\end{aligned}$$

Thus, we can clearly see that the overall bulk value of the Euler characteristic is the sum of the two individual values due to $SU(2)_+$ and $SU(2)_-$ gauge fields, giving:

$$\chi_{bulk} = \chi^+_{bulk} + \chi^-_{bulk} = 1 + 1 = 2 \tag{3.2.3.42}$$

This also shows both gauge fields contributing eqally to the overall Euler invariant.

Hirzebruch signature

Now we turn our attention to the other topological invariant, the Hirzebruch signature $\tau(M)$:

$$\tau(M) = -\frac{1}{24\pi^2}\left(\int_M \text{Tr } R\wedge R + \int_{\partial M} \text{Tr } \theta\wedge R + \eta_S(\partial M)\right) \tag{3.2.3.43}$$

The bulk part of the integral (3.2.3.43) can be given as:

$$\tau_{bulk} = -\frac{1}{24\pi^2}\int_M \text{Tr } R\wedge R = -\frac{1}{24\pi^2}\int_M R_{ab}\wedge R^{ab}$$

However, we can see from (3.2.3.5) that every element of the curvature 2-forms has a single 2-form term. Thus, we can write:

$$R_{ab}\wedge R^{ab} = 0 \qquad \Rightarrow \qquad \tau_{bulk} = 0$$

Now, as in the case of $\chi(M)$, the boundary integral term also vanishes following the same logic.

$$\theta_{ab}\wedge R^{ab} = 0 \qquad \Rightarrow \qquad \tau_{boundary} = 0$$

This leaves us with nothing but the last term, known as the spectral asymmetry term $\eta_S(\partial M)$ which in this case is also known to vanish. Therefore:

$$\tau(M) = 0 \tag{3.2.3.44}$$

As before, analyzing from the point of view of $SU(2)_\pm$ gauge fields lets us use:

$$\delta_{\mu\lambda}\delta_{\nu\gamma} - \delta_{\mu\gamma}\delta_{\nu\lambda} = \frac{1}{2}\left(\eta^{(+)i}_{\mu\nu}\eta^{(+)i}_{\lambda\gamma} + \eta^{(-)i}_{\mu\nu}\eta^{(-)i}_{\lambda\gamma}\right)$$

to write the bulk part of the signature complex as

$$\tau_{bulk} = -\frac{1}{24\pi^2}\int_M Tr\,(R\wedge R) = -\frac{2}{3}\left(\chi^+_{bulk} - \chi^-_{bulk}\right) = \frac{2}{3}\,(-1+1) = 0$$

where we can see that the individual bulk contribution is:

$$\begin{gathered}\tau_{bulk} = \tau^+_{bulk} + \tau^-_{bulk} \\ \tau^+_{bulk} = -\frac{2}{3}\chi^+_{bulk} = -\frac{2}{3} \qquad \tau^-_{bulk} = \frac{2}{3}\chi^-_{bulk} = \frac{2}{3}\end{gathered} \tag{3.2.3.45}$$

which concludes our computation of topological invariants of the Euclidean Schwarzschild metric.

3.3 Reduction of the generalized Darboux-Halphen system

Ablowitz et al [67, 90] studied the reduction of the SDYM equation with an infinite-dimensional Lie algebra to a 3×3 matrix differential equation. This led to a generalized Darboux-Halphen (gDH) system which differs from the DH system by a common additive term. The gDH system was also originally solved by Halphen [91] in terms of general hypergeometric functions and whose general solution admits movable natural barriers which can be densely branched.

Here, we discuss certain aspects related to the integrability of the gDH system. Some of these features were implicit in the original formulation of the system but were never made concrete. Specifically, we show that it is possible to derive naturally from the gDH system yet another reduced system of equations which satisfy a constraint. This constrained system resembles a non-autonomous Euler equation similar to that derived by Dubrovin [92] but with non-homogeneous terms. Furthermore, we derive a simple Lax pair for the constrained system.

3.3.1 The gDH system

In this subsection, we introduce the gDH system for the complex functions $\omega_i(t)$

$$\dot{\omega}_i = \omega_j\omega_k - \omega_i\left(\omega_j + \omega_k\right) + \tau^2\,, \quad i \neq j \neq k = 1, 2, 3,\ cyclic\,. \tag{3.3.1.1}$$

The common additive term τ^2 is elaborated as

$$\begin{aligned} \tau^2 =& \alpha_1^2 x_2 x_3 + \alpha_2^2 x_3 x_1 + \alpha_3^2 x_1 x_2 \qquad \text{with} \qquad x_i = \omega_j - \omega_k\,, \\ & i \neq j \neq k,\ cyclic\,, \qquad x_1 + x_2 + x_3 = 0\,, \end{aligned} \tag{3.3.1.2}$$

where $\alpha_i\,,\ i = 1, 2, 3$ are complex constants. As mentioned in Section 1, the gDH system arises from a particular reduction of the SDYM equations [67, 90]. They also appear in the study of $SU(2)$-invariant, hypercomplex four-manifolds [93]. In Section 3, we will provide a derivation of the gDH system from the SDYM reductions following [67].

In the following, we derive from (3.3.1.1) a reduced system of differential equations which satisfy a constraint.

Constrained gDH system

Note that the variables x_i defined in (3.3.1.2) satisfy the equations

$$\dot{x}_i = -2\omega_i x_i\,, \quad i = 1, 2, 3\,, \tag{3.3.1.3}$$

which are obtained from (3.3.1.1) by taking the difference of the equations for ω_j and ω_k. Using (3.3.1.3), the gDH equations (3.3.1.1) can be re-expressed as follows:

$$\dot{\omega}_i - \frac{\omega_i}{2}\left(\frac{\dot{x}_j}{x_j} + \frac{\dot{x}_k}{x_k}\right) = \omega_j\omega_k + \tau^2\,.$$

Then by defining new variables $W_i\,,\ i = 1, 2, 3$ via

$$W_i := \frac{\omega_i}{\sqrt{x_j x_k}}\,, \quad i \neq j \neq k,\ cyclic\,, \tag{3.3.1.4}$$

one obtains the system

$$\dot{W}_i = x_i W_j W_k + \frac{\tau^2}{\sqrt{x_j x_k}}\,. \tag{3.3.1.5}$$

It follows from (3.3.1.5) that

$$\sum_{i=1}^{3} W_i\dot{W}_i = W_1W_2W_3\sum_{i=1}^{3} x_i - \frac{\tau^2}{2x_1x_2x_3}\sum_{i=1}^{3}\dot{x}_i = 0$$

after using (3.3.1.4), (3.3.1.3) and the fact that $x_1+x_2+x_3=0$. Thus, one finds that the quantity

$$Q := \sum_{i=1}^{3} W_i^2 = \frac{\omega_1^2}{x_2x_3} + \frac{\omega_2^2}{x_1x_3} + \frac{\omega_3^2}{x_1x_2}$$

is a constant. However, the quantity Q is *not* a conserved quantity of (3.3.1.5), rather $Q=-1$ is an identity which follows from the definition of the variables W_i in (3.3.1.4). Indeed, a direct calculation using $x_1+x_2+x_3=0$, shows that

$$\begin{aligned} Q &= \frac{\omega_1^2x_1+\omega_2^2x_2+\omega_3^2x_3}{x_1x_2x_3} = \frac{\omega_1^2x_1+\omega_2^2x_2-\omega_3^2(x_1+x_2)}{x_1x_2x_3} \\ &= \frac{x_1(\omega_1-\omega_3)(\omega_1+\omega_3)+x_2(\omega_2-\omega_3)(\omega_2+\omega_3)}{x_1x_2x_3} = \frac{x_1x_2(\omega_2-\omega_1)}{x_1x_2x_3} = -\frac{x_1x_2x_3}{x_1x_2x_3} = -1 \end{aligned}$$

Therefore, the system in (3.3.1.5) is a reduction of the original gDH system; the reduced system can be regarded as a third order system for the W_i satisfying the constraint $Q=-1$. Note that the DH equations (3.4.1) being a special case ($\alpha_i=0$) of (3.3.1.1), also admits the same reduced system (3.3.1.5) as above but with $\tau=0$.

Remark: A third order system similar to (3.3.1.5) but without the non-homogeneous term, was introduced in [94, 95] where the authors derived a family of self-dual, SU(2)-invariant, Bianchi-IX metrics obtained from solutions of a special Painlevé-VI equation. In that case, the vanishing of the anti-self-dual Weyl tensor and scalar curvature led to a sixth order system described by the classical DH system (3.4.1) coupled to another third order system. There, the W_i variables represented different quantities although defined in the same way as in (3.3.1.4). The quantity Q was a first integral (instead of a number) in that case, depending on the initial conditions for the sixth order system. This sixth order system considered in [94, 95] also admits a special reduction to the third order DH system when the metric is self-dual Einstein. It is this latter case which corresponds to the homogeneous version of (3.3.1.5) above with $Q=-1$.

Next, we discuss the solution of the reduced system via the solutions of the original gDH system (3.3.1.1).

Solutions

As mentioned in the Introduction, Halphen [91] solved the gDH system and expressed its solution in terms of the general hypergeometric equation. Below we discuss a method of solution first given by Brioschi [96].

Let us first introduce a function $s(t)$ via the following ratio:

$$s = \frac{\omega_3-\omega_2}{\omega_1-\omega_2} = -\frac{x_1}{x_3}. \tag{3.3.1.6}$$

Taking the derivative of $\ln s$ in (3.3.1.6) and then using (3.3.1.3), the x_i can be written as

$$x_1 = -\frac{1}{2}\frac{\dot{s}}{s-1}, \qquad x_2 = \frac{1}{2}\frac{\dot{s}}{s}, \qquad x_3 = \frac{1}{2}\frac{\dot{s}}{s(s-1)}. \tag{3.3.1.7}$$

Using (3.3.1.3) once more, the gDH variables ω_i can be expressed in terms of $s, \dot{s}$ and $\ddot{s}$ as

$$\omega_1 = -\frac{1}{2}\frac{d}{dt}\left[\ln\left(\frac{\dot{s}}{s-1}\right)\right], \qquad \omega_2 = -\frac{1}{2}\frac{d}{dt}\left[\ln\left(\frac{\dot{s}}{s}\right)\right], \qquad \omega_3 = -\frac{1}{2}\frac{d}{dt}\left[\ln\left(\frac{\dot{s}}{s(s-1)}\right)\right]. \tag{3.3.1.8}$$

Substituting the above expressions for ω_i into the gDH system (3.3.1.1) yields the following third order equation for $s(t)$

$$\frac{\dddot{s}}{\dot{s}} - \frac{3}{2}\left(\frac{\ddot{s}}{\dot{s}}\right)^2 + \frac{\dot{s}^2}{2}\left[\frac{1-\alpha_1^2}{s^2} + \frac{1-\alpha_2^2}{(s-1)^2} + \frac{\alpha_1^2+\alpha_2^2-\alpha_3^2-1}{s(s-1)}\right] = 0\,, \tag{3.3.1.9}$$

also known as the Schwarzian equation. Equation (3.3.1.9) can be *linearized* in terms of the hypergeometric equation as follows. Let $\chi_1(s)$ and $\chi_2(s)$ be any two linearly independent solution of the hypergeometric equation

$$\chi'' + \left(\frac{1-\alpha_1}{s} + \frac{1-\alpha_2}{s-1}\right)\chi' + \frac{(\alpha_1+\alpha_2-1)^2-\alpha_3^2}{4s(s-1)}\chi = 0\,. \tag{3.3.1.10}$$

If the independent variable t in the gDH system is defined by

$$t(s) = \frac{\chi_2(s)}{\chi_1(s)}\,, \tag{3.3.1.11}$$

then the inverse function $s(t)$ satisfies the Schwarzian equation above. Thus, it is possible to express the gDH variables ω_i in terms of the hypergeometric solution χ_1 and its derivative.

The reduced system (3.3.1.5) takes a simple but interesting form if we consider a variable change from t to s and re-express the corresponding equations. First, let us define new variables

$$\widehat{W_1} = \frac{W_1}{2}\,, \qquad \widehat{W_2} = \frac{W_2}{2i}\,, \qquad \widehat{W_3} = \frac{W_3}{2i}\,, \tag{3.3.1.12}$$

where $i := \sqrt{-1}$. Then by using the parametrization of the x_i from (3.3.1.7) in (3.3.1.5), one obtains a non-autonomous, non-homogeneous version of the Euler "top" equations, namely,

$$\widehat{W_1}' = \frac{\widehat{W_2}\widehat{W_3}}{s-1} + \frac{f(s)}{\sqrt{s-1}}\,, \qquad \widehat{W_2}' = \frac{\widehat{W_1}\widehat{W_3}}{s} - \frac{f(s)}{\sqrt{s}}\,, \qquad \widehat{W_3}' = \frac{\widehat{W_1}\widehat{W_2}}{s(s-1)} - \frac{f(s)}{\sqrt{s(s-1)}}\,,$$
$$\text{where} \qquad f(s) = \frac{\alpha_1^2(s-1)-\alpha_2^2 s-\alpha_3^2 s(s-1)}{4s(s-1)}\,, \tag{3.3.1.13}$$

and "prime" indicates derivative with respect to s. It follows from (3.3.1.13) that

$$\begin{aligned}&\widehat{W_1}\widehat{W_1}' - \widehat{W_2}\widehat{W_2}' - \widehat{W_3}\widehat{W_3}' \\ &= \widehat{W_1}\widehat{W_2}\widehat{W_3}\left(\frac{1}{s-1} - \frac{1}{s} - \frac{1}{s(s-1)}\right) + f(s)\left(\frac{\widehat{W_1}}{\sqrt{s-1}} + \frac{\widehat{W_2}}{\sqrt{s}} + \frac{\widehat{W_3}}{\sqrt{s(s-1)}}\right) = 0\,.\end{aligned}$$

The interested reader can easily verify using (3.3.1.12), (3.3.1.6) and (3.3.1.4) that the coefficient of $f(s)$ vanishes identically in above, thereby showing that $\widehat{W_1}^2 - \widehat{W_2}^2 - \widehat{W_3}^2$ is a constant. Moreover, from (3.3.1.12) one can easily compute

$$\gamma = \widehat{W_1}^2 - \widehat{W_2}^2 - \widehat{W_3}^2 = \frac{1}{4}\sum_{i=1}^{3} W_i^2 = \frac{1}{4}Q = -\frac{1}{4}\,.$$

Thus, the reduced system (3.3.1.13) for the $\widehat{W_i}$ satisfy the constraint $\gamma = -\frac{1}{4}$.

For for the DH case, $f(s)=0$ (because $\alpha_i=0$), then (3.3.1.13) becomes a homogeneous, non-autonomous Euler system that arises in hydrodynamic systems [92] as well as in self-dual Einstein equations for $SU(2)$-invariant Bianchi 1X metrics [94, 95, 97] (see Remark in Section 2.1). It is known that this homogeneous system can be solved in terms of a special Painlevé VI equation via a transformation discussed in [98], or from the Schlesinger equations associated with the Painlevé VI equation [97]. In general, the solution for the reduced system (3.3.1.13) can be expressed in terms of hypergeometric functions utilizing the transformation given by (3.3.1.11) and the parametrization of x_i and ω_i given in (3.3.1.7) and (3.3.1.8). One also uses the relation $\dot{s}=1/t'(s)=\chi_1^2/W$ where $\chi_1(s)$ is a solution of (3.3.1.10) and $W(s):=W(\chi_1,\chi_2)$ is the Wronskian of two independent solutions. Finally, taking into account the definitions from (3.3.1.4) and (3.3.1.12) the explicit form of the solutions are

$$\widehat{W}_1(s)=-\frac{s\sqrt{s-1}}{2}\left(2\frac{\chi_1'}{\chi_1}-\frac{W'}{W}-\frac{1}{s-1}\right),\qquad \widehat{W}_2(s)=\frac{\sqrt{s}(s-1)}{2}\left(2\frac{\chi_1'}{\chi_1}-\frac{W'}{W}-\frac{1}{s}\right),$$
$$\widehat{W}_3(s)=\frac{\sqrt{s(s-1)}}{2}\left(2\frac{\chi_1'}{\chi_1}-\frac{W'}{W}-\frac{1}{s}-\frac{1}{s-1}\right),$$

Moreover, applying Abel's formula to (3.3.1.10), W'/W is expressed as

$$\frac{W'}{W}=-\left(\frac{1-\alpha_1}{s}+\frac{1-\alpha_2}{s-1}\right).$$

A more direct way to solve the $\widehat{W}_i$ is to reduce the system (3.3.1.13) into a single, scalar ordinary differential equation for one of the variables. Recall that the $\widehat{W}_i$ satisfy the following constraints, namely,

$$\widehat{W}_1^{\,2}-\widehat{W}_2^{\,2}-\widehat{W}_3^{\,2}=-\frac{1}{4},\qquad \frac{\widehat{W}_1}{\sqrt{s-1}}+\frac{\widehat{W}_2}{\sqrt{s}}+\frac{\widehat{W}_3}{\sqrt{s(s-1)}}=0. \tag{3.3.1.14}$$

By regarding these constraints as two equations for the $\widehat{W}_i$, it is possible to solve for any two of them, say, $\widehat{W}_1$ and $\widehat{W}_3$ in terms of $\widehat{W}_2$. Thus, one obtains

$$\widehat{W}_1=\frac{c-\sqrt{s}\,\widehat{W}_2}{\sqrt{s-1}},\qquad \widehat{W}_3=\frac{\widehat{W}_2-c\sqrt{s}}{\sqrt{s-1}},\qquad c=\pm\frac{1}{2}. \tag{3.3.1.15}$$

Next, substituting the expressions for $\widehat{W}_1$ and $\widehat{W}_3$ from (3.3.1.15) into the equation for $\widehat{W}_2$ in (3.3.1.13), yields a Riccati equation

$$\sqrt{s}(s-1)\widehat{W}_2'+\widehat{W}_2^{\,2}-c\frac{s+1}{\sqrt{s}}\widehat{W}_2+(s-1)f(s)+\frac{1}{4}=0,$$

where the rational function $f(s)$ is given in (3.3.1.13). If we take $c=\frac{1}{2}$, then the Riccati equation can be linearized by the following transformation

$$\frac{\widehat{W}_2}{\sqrt{s}(s-1)}=\frac{1}{2}\Big(\frac{1-\alpha_2}{s-1}-\frac{\alpha_1}{s}\Big)+\frac{\chi'}{\chi}$$

where the function $\chi(s)$ satisfies the hypergeometric equation (3.3.1.10). If $c=-\frac{1}{2}$, then one can still linearize the resulting Riccati equation but the parameters in the underlying hypergeometric equation are related to but are not the same as the α_i.

3.3.2 The DH-IX matrix system

So far we have dealt with the gDH system which consists of the DH equations together with a common additive term τ^2 appearing in all three equations in (3.3.1.1). In this subsection, we will show how the gDH system can be derived from a 3×3 matrix system which arise as a reduction of the SDYM field equations. We start by reviewing the reduction process on the SDYM equations following [67].

Consider a gauge group G which may be a finite or infinite-dimensional Lie group. The gauge field F is a 2-form taking values in the associated Lie algebra $\mathfrak{g}$, and is given in terms of the $\mathfrak{g}$-valued connection 1-form (gauge potential) A as $F = dA - A \wedge A$. In a local co-ordinate system $\{x^a\}$ $a = 0, 1, 2, 3$ the gauge field components are given by $F_{ab} = \partial_a A_b - \partial_b A_a - [A_a, A_b]$ where ∂_a denotes partial derivative with respect to x^a and $[\, ,\,]$ denotes the Lie bracket in $\mathfrak{g}$. The self-duality condition implies that $F = {}^*F$ where ${}^*F$ is the dual 2-form. In terms of components of F, the self-duality condition is equivalent to

$$F_{0i} = F_{jk}\,, \qquad i \neq j \neq k\,,\ cyclic\,. \tag{3.3.2.1}$$

If the connection 1-form is restricted to depend only on the co-ordinate $x^0 := t$, then without loss of generality, one can choose a gauge where $A_0 = 0$. Consequently, $A_i = A_i(t)$ for $i = 1, 2, 3$, and the resulting SDYM equations (3.3.2.1) becomes the Nahm equations [99]

$$\dot{A}_i = [A_j, A_k]\ , \qquad i \neq j \neq k\,,\ cyclic\,. \tag{3.3.2.2}$$

Suppose the Lie algebra $\mathfrak{g}$ is chosen to be $\mathfrak{sdiff}(S^3)$ - the infinite-dimensional Lie algebra of diffeomorphisms on S^3 generated by the left-invariant vector fields X_i satisfying the relation $[X_i, X_j] = X_k,\ i \neq j \neq k\,,\ cyclic\,$. Furthermore, let the A_i be of the form

$$A_i = -\sum_{j,k=1}^{3} M_{ij}(t) O_{jk} X_k \tag{3.3.2.3}$$

where $M(t)$ is a 3×3 matrix with entries M_{ij} and $O_{ij} \in SO(3)$ represents a point on S^3. Then the Nahm equations (3.3.2.2) lead to the following matrix ordinary differential equation for $M(t)$ [67, 93]

$$\dot{M} = C(M) + M^T M - (\mathrm{Tr} M) M\,, \tag{3.3.2.4}$$

where $C(M)$ denotes the matrix of cofactors of M. Equation (3.3.2.4) is a ninth-order coupled system of equations for the matrix elements of $M(t)$ and was referred to as the DH-IX system in [90, 67]. Indeed, by expressing the matrix elements of M as

$$M = \begin{pmatrix} \Omega_1 & \theta_3 & \phi_2 \\ \phi_3 & \Omega_2 & \theta_1 \\ \theta_2 & \phi_1 & \Omega_3 \end{pmatrix}$$

the component equations in (3.3.2.4) can be explicitly written out as

$$\begin{aligned} \dot{\Omega}_i &= \Omega_j \Omega_k - \Omega_i(\Omega_j + \Omega_k) - \theta_i \phi_i + \theta_j^2 + \phi_k^2 \\ \dot{\theta}_i &= -\left(\theta_i + \phi_i\right) \Omega_i - \left(\theta_i - \phi_i\right) \Omega_k + \theta_k \left(\theta_j + \phi_j\right) \\ \dot{\phi}_i &= -\left(\theta_i + \phi_i\right) \Omega_i + \left(\theta_i - \phi_i\right) \Omega_j + \phi_j \left(\theta_k + \phi_k\right)\ , \end{aligned} \tag{3.3.2.5}$$

$i \neq j \neq k = 1, 2, 3$, *cyclic*. Equations (3.3.2.5) can be regarded as the original DH system but with individual additive terms. We next show how to recast the DH-IX equation into the gDH system (3.3.1.1) where the equations have a common additive term.

Reduction of DH-IX to the gDH system

Note that the equations for the off-diagonal entries in (3.3.2.5) involve symmetric and skew-symmetric combinations of the off-diagonal elements. This fact can be exploited further to simplify the matrix equation (3.3.2.4) as follows: First, the cofactor matrix $C(M) = (\mathrm{adj}M)^T$, where the adjoint matrix can be expressed as

$$\mathrm{adj}M = M^2 - (\mathrm{Tr}M)M + \frac{1}{2}\Big((\mathrm{Tr}M)^2 - \mathrm{Tr}M^2\Big)I$$

using the Caley-Hamilton theorem for 3×3 matrices. In above, Tr denotes the matrix trace and I is the identity matrix. Next, substituting the transpose of the above expression for $C(M)$ into (3.3.2.4) yields

$$\dot{M} = (M^T - (\mathrm{Tr}M)I)(M + M^T) + \frac{1}{2}\Big((\mathrm{Tr}M)^2 - \mathrm{Tr}M^2\Big)I\,. \tag{3.3.2.6}$$

Equation (3.3.2.6) motivates decomposing the matrix M into its symmetric and skew-symmetric parts and re-expressing the DH-IX system in terms of these components as illustrated below. Let us consider the following decomposition of M

$$M = M_s + M_a = P(d+a)P^{-1}\,, \tag{3.3.2.7}$$

where the symmetric part M_s is further diagonalized by a orthogonal matrix P ($P^T = P^{-1}$) and the skew-symmetric part is expressed as $M_a = PaP^{-1}$ with

$$d = \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix}, \qquad a = \begin{pmatrix} 0 & \tau_3 & -\tau_2 \\ -\tau_3 & 0 & \tau_1 \\ \tau_2 & -\tau_1 & 0 \end{pmatrix}. \tag{3.3.2.8}$$

Substituting (3.3.2.7) into (3.3.2.6) yields the following set of matrix equations for P, a and d,

$$\dot{P}+Pa = 0\,, \qquad \dot{a}+ad+da = 0\,, \qquad \dot{d} = 2d^2-2(\mathrm{Tr}d)d+\frac{1}{2}\Big(\mathrm{Tr}d^2-(\mathrm{Tr}d)^2-2\mathrm{Tr}a^2\Big)I \tag{3.3.2.9}$$

The last equation of (3.3.2.9) gives the gDH system (3.3.1.1) with $\tau^2 = \tau_1^2 + \tau_2^2 + \tau_3^2$. Then, using (3.3.1.3) one can integrate the second equation in (3.3.2.9), i.e.,

$$\dot{\tau}_i = -\tau_i(\omega_j + \omega_k) \quad \Rightarrow \quad \tau_i^2 = \alpha_i^2 x_j x_k = \alpha_i^2\,(\omega_j - \omega_i)\,(\omega_i - \omega_k)\;, \quad i \neq j \neq k,\ cyclic\,, \tag{3.3.2.10}$$

$$\tau_1 = \kappa_1 \frac{\dot{s}}{\sqrt{s(s-1)}} \qquad \tau_2 = \kappa_2 \frac{\dot{s}}{s\sqrt{(s-1)}} \qquad \tau_3 = \kappa_3 \frac{\dot{s}}{\sqrt{s}(s-1)}, \tag{3.3.2.11}$$

and where α_i are integration constants. Combining the last equation of (3.3.2.9) with (3.3.2.10), yields the gDH system (3.3.1.1). The first equation in (3.3.2.9) is linear and can be solved for P given the τ_i although it is not possible to obtain closed form solutions for P except for special cases. We illustrate one such special case in the example below.

The DH-V system

We now discuss a fifth order reduction of the DH-IX system where the matrix P introduced in (3.3.2.7) can be expressed in closed form. Let us consider the case in which the DH-IX matrix has the special form

$$M = \begin{pmatrix} \Omega_1 & \theta & 0 \\ \phi & \Omega_2 & 0 \\ 0 & 0 & \Omega_3 \end{pmatrix}.$$

Then (3.3.2.5) becomes a fifth-order system given by

$$\begin{aligned}
\dot{\Omega}_1 &= \Omega_2\Omega_3 - \Omega_1(\Omega_2+\Omega_3) + \phi^2 \\
\dot{\Omega}_2 &= \Omega_3\Omega_1 - \Omega_2(\Omega_3+\Omega_1) + \theta^2 \\
\dot{\Omega}_3 &= \Omega_1\Omega_2 - \Omega_3(\Omega_1+\Omega_2) - \theta\phi \\
\dot{\theta} &= -(\theta+\phi)\,\Omega_3 - (\theta-\phi)\,\Omega_2 \\
\dot{\phi} &= -(\theta+\phi)\,\Omega_3 + (\theta-\phi)\,\Omega_1
\end{aligned} \tag{3.3.2.12}$$

which was introduced in [100]. We refer to system (3.3.2.12) as the DH-V system and will construct its solution based on the method discussed in Section 3.1. Due to the special block structure of M, its symmetric part M_s can be diagonalized by an orthogonal matrix of the form

$$P = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.3.2.13}$$

where $\gamma = \gamma(t)$ is a complex function to be determined. That is, $M_s = PdP^{-1}$ with d as in (3.3.2.8). Furthermore, the skew-symmetric part M_a commutes with the P above so that

$$a = P^{-1}M_aP = M_a = \begin{pmatrix} 0 & \tau_3 & 0 \\ -\tau_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \tau_3 = \frac{1}{2}(\theta-\phi) \tag{3.3.2.14}$$

Since $\tau_1 = \tau_2 = 0$ for the DH-V case, we have $\tau^2 = \tau_3^2$ in (3.3.1.1) which is now solved via the Schwarzian equation (3.3.1.9) with $\alpha_1 = \alpha_2 = 0$. Moreover, using (3.3.2.10) and (3.3.1.7), one obtains

$$\tau_3 = \alpha_3\sqrt{x_1x_2} = \pm\frac{i\alpha_3}{2}\frac{\dot{s}}{\sqrt{s(s-1)}}\,. \tag{3.3.2.15}$$

Then the first equation in (3.3.2.9),

$$\dot{P} = -Pa \qquad \Rightarrow \qquad \dot{\gamma} = \tau_3\,,$$

which can be solved in terms of $s(t)$ as

$$\gamma(s(t)) = \pm i\alpha_3\log\left(\sqrt{s}-\sqrt{s-1}\right) + \gamma_0\,, \tag{3.3.2.16}$$

where γ_0 is a (complex) constant. Hence the DH-V matrix M can be reconstructed in terms of the matrices P, d and a as follows:

$$\begin{aligned}
\Omega_1+\Omega_2 &= \omega_1+\omega_2\,, \qquad \Omega_1-\Omega_2 = (\omega_1-\omega_2)\cos 2\gamma\,, \qquad \Omega_3 = \omega_3\,, \\
\theta+\phi &= (\omega_2-\omega_1)\sin 2\gamma\,, \qquad \theta-\phi = 2\tau_3\,,
\end{aligned} \tag{3.3.2.17}$$

where ω_i are given by (3.3.1.8), and τ_3, γ are given by equations (3.3.2.15) and (3.3.2.16), respectively. Equation (3.3.2.17) gives the complete solution of the DH-V system in terms of the solution $s(t)$ of Schwarzian equation (3.3.1.9) with $\alpha_1 = \alpha_2 = 0$.

It is also possible to express the constraint Q introduced in Section 2.1 in terms of the DH-V matrix elements. Indeed, one can calculate from (3.3.2.17)

$$\begin{aligned}
\omega_1 &= \frac{1}{2}\left(\Sigma+\Delta\right)\,, \qquad \omega_2 = \frac{1}{2}\left(\Sigma-\Delta\right)\,, \qquad \omega_3 = \Omega_3\,, \\
\Sigma &:= \Omega_1+\Omega_2\,, \qquad \Delta := \pm\sqrt{(\Omega_1-\Omega_2)^2+(\theta+\phi)^2}\,.
\end{aligned} \tag{3.3.2.18}$$

Substituting these expressions into the definition of Q, yields

$$
\begin{aligned}
Q &:= \frac{\omega_1^2}{x_2x_3} + \frac{\omega_2^2}{x_3x_1} + \frac{\omega_3^2}{x_1x_2} \\
&= \frac{\frac{1}{4}\left(\Sigma+\Delta\right)^2\left(\frac{1}{2}(\Sigma-\Delta)-\Omega_3\right) - \frac{1}{4}\left(\Sigma-\Delta\right)^2\left(\frac{1}{2}(\Sigma+\Delta)-\Omega_3\right) + \Omega_3^2\Delta}{\Delta\left(\Omega_3-\frac{1}{2}(\Sigma+\Delta)\right)\left(\frac{1}{2}(\Sigma-\Delta)-\Omega_3\right)} = -1
\end{aligned}
$$

after some simplification.

3.3.3 Lax pair and Hamiltonian for the constrained gDH system

In this subsection we derive a Lax pair for the reduced non-homogeneous system (3.3.1.13) for the $\widehat{W}_i$ introduced in Section **3.3.1**. Furthermore, we show that (3.3.1.13) can also be regarded as a constrained Hamiltonian system in the phase space of the variables $\widehat{W}_i$.

Lax equation

Specifically, we find 3×3 matrices U and V such that (3.3.1.13) is equivalent to the following matrix Lax equation

$$
U' + [U,V] = 0\,,
$$

where recall that "prime" denotes $\frac{d}{ds}$. Let us choose U and V in the Lie algebra $so(1,2)$ as follows:

$$
U = \begin{pmatrix} 0 & \widehat{W}_3 & \widehat{W}_2 \\ \widehat{W}_3 & 0 & \widehat{W}_1 \\ \widehat{W}_2 & -\widehat{W}_1 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & v_3 & v_2 \\ v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{pmatrix}, \tag{3.3.3.1}
$$

where the v_i are to be determined. The commutator $[U,V]$ is also in $so(2,1)$ and its entries should be equal to the right hand side of (3.3.1.13), which we denote by r_i, $i=1,2,3$. This results in the following linear system

$$
B\boldsymbol{v} = \boldsymbol{r}\,, \qquad B = \begin{pmatrix} 0 & -\widehat{W}_3 & \widehat{W}_2 \\ -\widehat{W}_3 & 0 & \widehat{W}_1 \\ \widehat{W}_2 & -\widehat{W}_1 & 0 \end{pmatrix}, \qquad \boldsymbol{v} = [v_1,v_2,v_3]^T\,, \quad \boldsymbol{r} = [r_1,r_2,r_3]^T\,. \tag{3.3.3.2}
$$

for the vector $\boldsymbol{v}$. Note that the matrix B is singular. In order for the linear system (3.3.3.2) to have a *consistent* solution, the vector $\boldsymbol{r}$ must be orthogonal to the null space of B^T by Fredholm's alternative. The null space of B^T is spanned by the vector $\widehat{\boldsymbol{N}} = [\widehat{W}_1, -\widehat{W}_2, -\widehat{W}_3]^T$. Therefore, one must have $\widehat{\boldsymbol{N}}\cdot\boldsymbol{r} = \widehat{W}_1 r_1 - \widehat{W}_2 r_2 - \widehat{W}_3 r_3 = 0$, which is readily verified from the calculations following (3.3.1.13). Thus, the linear system (3.3.3.2) admits infinitely many solutions (defined modulo the homogeneous solution spanned by the null vector $[\widehat{W}_1, \widehat{W}_2, \widehat{W}_3]^T$ of B). A particular choice for the vector $\boldsymbol{v}$ is given by

$$
v_1 = 0\,, \qquad v_2 = -\frac{r_3}{\widehat{W}_1} = -\left(\frac{\widehat{W}_2}{s(s-1)} + \frac{f(s)}{\widehat{W}_1\sqrt{s(s-1)}}\right), \qquad v_3 = \frac{r_2}{\widehat{W}_1} = \left(\frac{\widehat{W}_3}{s} + \frac{f(s)}{\widehat{W}_1\sqrt{s}}\right),
$$

which then yields the matrix V in the Lax pair.

In a general setting, the Lax equation $U' + [U, V] = 0$ is useful to generate a sequence of conserved quantities $\mathrm{Tr}U^n$, $n = 1, 2, \ldots$. Indeed, by differentiating with respect to s one obtains

$$\left(\mathrm{Tr}U^n\right)' = n\mathrm{Tr}\left(U^{n-1}[V, U]\right) = n\mathrm{Tr}\left(V[U, U^{n-1}]\right) = 0\,.$$

These conserved quantities are related to the symmetric functions of the eigenvalues of of the matrix U. In the present case, the eigenvalues of U are simply given by $\lambda = 0, \pm\sqrt{-\gamma} = 0, \pm\frac{1}{2}$ since $\gamma = \frac{1}{4}Q = -\frac{1}{4}$. In fact, one obtains $\mathrm{Tr}U = 0, \mathrm{Tr}U^2 = -2\gamma$, and the remaining traces which are polynomials in γ can be calculated by applying the Caley-Hamilton theorem.

It is worth pointing out that (3.3.1.5) for the W_i also admits a Lax pair. Here, one chooses $so(3)$-valued 3×3 matrices

$$L = \begin{pmatrix} 0 & W_3 & -W_2 \\ -W_3 & 0 & W_1 \\ W_2 & -W_1 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & A_3 & -A_2 \\ -A_3 & 0 & A_1 \\ A_2 & -A_1 & 0 \end{pmatrix}, \tag{3.3.3.3}$$

where the $W_i(t)$ are defined in (3.3.1.4) and the $A(t)$ are to be determined such that the Lax equation $\dot{L} + [L, A] = 0$ is equivalent to the system (3.3.1.5). The matrix A can be found by proceeding in a similar fashion as outlined above. One finds that a particular choice for the matrix elements of A is given by

$$A_1 = 0\,, \qquad A_2 = x_3W_2 + \frac{\tau^2}{W_1\sqrt{x_1x_2}}\,, \qquad A_3 = -\left(x_2W_3 + \frac{\tau^2}{W_1\sqrt{x_1x_3}}\right).$$

The eigenvalues of L is given by $\lambda = 0, \pm\sqrt{-Q} = 0, \pm 1$. Consequently, $\mathrm{Tr}L^n$, $n = 1, 2, \ldots$ are polynomials in Q.

Hamiltonian formulation

Equations (3.3.1.13) can be also regarded as a constrained Hamiltonian system in the phase space of the variables $\widehat{W}_i$ satisfying the constraints in (3.3.1.14). The phase space is endowed with a natural Poisson structure inherited from the Lie-Poisson structure defined on the dual space of the Lie algebra $so(1, 2)$ used to construct the Lax pair. Explicitly, the Poisson structure is given by the fundamental Poisson bracket relations

$$\{\widehat{W}_1, \widehat{W}_2\} = \widehat{W}_3\,, \qquad \{\widehat{W}_2, \widehat{W}_3\} = -\widehat{W}_1\,, \qquad \{\widehat{W}_3, \widehat{W}_1\} = \widehat{W}_2\,. \tag{3.3.3.4}$$

In general, the Poisson bracket of any two continuously differentiable functions f and g on the phase space, is given by

$$\{f, g\} = J(df, dg) = \sum_{i,j,k=1}^{3} C_{ij}^k \widehat{W}_k \frac{\partial f}{\partial \widehat{W}_i}\frac{\partial f}{\partial \widehat{W}_j}\,,$$

where C_{ij}^k are the structure constants for the Lie algebra $so(1, 2)$. The Poisson tensor $J_{ij} := \sum_{k=1}^{3} C_{ij}^k \widehat{W}_k$ is degenerate on the three-dimensional phase space, and admits a Casimir function constructed from the Lax matrix U as follows

$$C = -\frac{1}{2}\mathrm{Tr}U^2 = \widehat{W}_1^{\,2} - \widehat{W}_2^{\,2} - \widehat{W}_3^{\,2}$$

such that $J(\cdot, dC) = 0$. In other words, $\{f, C\} = 0$ for any smooth function f on the phase space. Note from (3.3.1.14) that $C + \frac{1}{4} = 0$ is one of the constraints, while the other constraint is given by $l = 0$, where

$$l = -\frac{1}{2}\mathrm{Tr}US = \frac{\widehat{W}_1}{\sqrt{s-1}} + \frac{\widehat{W}_2}{\sqrt{s}} + \frac{\widehat{W}_3}{\sqrt{s(s-1)}}$$

$$S = \begin{pmatrix} 0 & (s(s-1))^{-1/2} & s^{-1/2} \\ (s(s-1))^{-1/2} & 0 & (s-1)^{-1/2} \\ s^{-1/2} & -(s-1)^{-1/2} & 0 \end{pmatrix} .$$

Next, we introduce a Hamiltonian function on the phase space by

$$H = -\frac{1}{2}\mathrm{Tr}(UIU - 4cf(s)US) = \frac{1}{2}\left(\frac{\widehat{W}_1^{\,2}}{s-1} - \frac{\widehat{W}_2^{\,2}}{s} - \frac{(2s-1)\widehat{W_3^2}}{s(s-1)}\right) - 4cf(s)l\,, \qquad (3.3.3.5)$$

where $I = \mathrm{diag}(s^{-1}, (s-1)^{-1}, 0)$, l is defined above, $c = \pm\frac{1}{2}$, and $f(s)$ is defined in (3.3.1.13). With the fundamental Poisson brackets given by (3.3.3.4), the reduced gDH equations (3.3.1.13) can be expressed by the following equation of motions together with the constraints

$$\dot{\widehat{W}}_i = \{\widehat{W}_i, H\}\,, \qquad C + \frac{1}{4} = 0\,, \qquad l = 0\,, \qquad (3.3.3.6)$$

where the Hamiltonian H is given by (3.3.3.5). The equations of of motions obtained from (3.3.3.6) determines the equations in (3.3.1.13) after applying the constraints. For example, one can compute using (3.3.3.4) that

$$\{\widehat{W}_1, H\} = \frac{\widehat{W}_2\widehat{W}_3}{s-1} - 4cf(s)\left(\frac{\widehat{W}_2}{\sqrt{s(s-1)}} - \frac{\widehat{W}_3}{\sqrt{s}}\right) .$$

Upon applying the constraints, one can replace $\widehat{W}_3$ in the second term above by its expression from (3.3.1.15) to obtain the first equation in (3.3.1.13). The remaining equations in (3.3.3.6) lead to the corresponding equations in (3.3.1.13) in a similar fashion.

For consistency, it also needs to be checked that the constraints are satisfied by the Hamiltonian dynamics. In other words, one should have modulo the constraints

$$\frac{dC}{ds} = \{C, H\} = 0\,, \qquad \frac{dl}{ds} = \frac{\partial l}{\partial s} + \{l, H\} = 0\,.$$

The first consistency condition is obviously satisfied since C is a Casimir function, the second one can also be verified by using (3.3.1.15) and after some straightforward computations.

3.4 Bianchi-IX, Darboux-Halphen and Chazy-Ramanujan

[1] Following applications to homogeneous cosmology, $\mathcal{M}_4$ spaces topologically equivalent to $\mathbb{R} \times \mathcal{M}_3$ of Bianchi type have been extensively explored. For a pedagogical introduction

[1]Near the completion of the article related to this work, the paper [101] appeared on arXiv. The authors, among other things, also address the question of arithmetics and integrability of Bianchi IX gravitational instantons, which have been explored by imposing the self-duality condition on triaxial Bianchi IX metrics and by employing a time-dependent conformal factor. We comment more about this in the final section at the end of the paper.

to various cosmological models look at the Lecture Notes by Ellis et. al.[102]. Out of all Bianchi type models of classes I - IX with vanishing cosmological constant, only Bianchi-IX has been found to exhibit a relationship with quasimodular forms [103, 104]. Modular forms in physics are a consequence of duality properties, resulting either from an invariance or a relationship between two distinct theories. In the past 30 years, modular and quasimodular forms have emerged mostly in the study of gravity and string theory [105]. Furthermore, we must note that the Bianchi-IX model is a controversial system (possessing both integrable and non-integrable aspects).

Various authors debated their doubts over statements about the chaotic nature of Bianchi-IX dynamics, simultaneously expressing their opinion that the model might be a classical integrable system (in Liouville sense) [106]. Thus, the Bianchi-IX cosmological model is good for testing theories in order to understand various concepts of integrability.

The generalized Darboux-Halphen system has been heavily studied over the recent years, the relationship between its classical form and the Bianchi-IX metric being established in the 90s. This connection proves useful in constructing various interesting Bianchi IX solutions [107] or their applications e.g. in the study of scattering of $SU(2)$ BPS monopoles [64, 108]. The Darboux-Halphen system also exhibits Ricci flow that describes the evolution of $SU(2)$-homogeneous 3D geometries and can be seen as reflection of hidden symmetrry of hyperbolic monopole motion [109].

The Darboux-Halphen differential equations, often called the classical Darboux-Halphen (DH) system

$$\dot{\omega}_i = \omega_j\omega_k - \omega_i\left(\omega_j + \omega_k\right), \qquad i \neq j \neq k = 1, 2, 3, \ cyclic, \qquad \dot{y} := \frac{dy}{dt}, \tag{3.4.1}$$

was originally formulated by Darboux [110] and subsequently solved by Halphen [111]. The general solution to equation (3.4.1) may be expressed in terms of the elliptic modular function. In fact Halphen related the DH equation with the null theta functions.

The system (3.4.1) has found applications in mathematical physics in relation to magnetic monopole dynamics [112], self dual Einstein equations [113, 97], topological field theory [114] and reduction of self-dual Yang-Mills (SDYM) equations [115]. Recently in [116], the DH system was reviewed from the perspective of the self-dual Bianchi-IX metric and the SDYM field equations, describing a gravitational instanton in the former case, and a Yang-Mills instanton in the latter. All systems related to the DH system such as Ramanujan and Ramamani system were covered, as well as aspects of integrability of the DH system.

The Bianchi-IX metric is a general setup for 4D Euclidean spherically symmetric metrics. Under certain settings of its curvature-wise anti-self-dual case, becomes the Taub-NUT. Naturally, the analysis of its connection and curvature follows the same way as in [117, 37].

3.4.1 Geometric analysis

The Bianchi-IX metric is written as:

$$ds^2 = \left[c_1(r)c_2(r)c_3(r)\right]^2 dr^2 + c_1^2(r)\sigma_1^2 + c_2^2(r)\sigma_2^2 + c_3^2(r)\sigma_3^2 \tag{3.4.1.1}$$

where the variables σ_i obey the structure equation:

$$d\sigma^i = -\varepsilon^i{}_{jk}\, \sigma^j \wedge \sigma^k \qquad \text{where} \qquad \sigma^i = -\frac{1}{r^2}\eta^i_{\mu\nu}x^\mu dx^\nu \tag{3.4.1.2}$$

and i, j, k are permutations of the indices $1, 2, 3$, with the t'Hooft symbols defined by Appendix 6.1.
Those solutions that are (anti-) self-dual fall into 2 categories:

1. connection wise self-dual
2. curvature wise self-dual

We shall now uncover the systems characterizing each category respectively.

Connection wise self duality - the Lagrange system

First we shall compute the spin connections of (3.4.1.1) from the tetrads [37]. We can list the tetrads of (3.4.1.1) as:

$$e^0 = c_0(r)dr \qquad e^i = c_i(r)\sigma^i, \qquad \text{where} \quad c_0(r) = c_1(r)c_2(r)c_3(r), \quad i = 1, 2, 3. \tag{3.4.1.3}$$

Obviously, e^0 produces no connections ($de^0 = 0$). However, for the remaining three, under torsion-free condition the 1st Cartan structure equation ($de^i = -\omega^i{}_j \wedge e^j$) from (3.4.1.3) gives us the following spin connections:

$$de^i = -\frac{\partial_r c_i}{c_0}\,\sigma^i \wedge e^0 - \left\{-\varepsilon^i{}_{jk}\frac{c_i^2 + c_j^2 - c_k^2}{2c_ic_j}\,\sigma^k \wedge e^j - \varepsilon^i{}_{kj}\frac{c_i^2 + c_k^2 - c_j^2}{2c_ic_k}\,\sigma^j \wedge e^k\right\},$$

$$\omega^i{}_0 = \frac{\partial_r c_i}{c_0}\,\sigma^i \qquad\qquad \omega^i{}_j = -\varepsilon^i{}_{jk}\frac{c_i^2 + c_j^2 - c_k^2}{2c_ic_j}\,\sigma^k \tag{3.4.1.4}$$

This elaborate form for the components of the spin connections make its anti-symmetric nature evident. If we only consider (anti-) self-dual cases of (3.4.1.4), we will have:

$$\omega_{0i} = \pm\frac{1}{2}\varepsilon_{0i}{}^{jk}\omega_{jk} = \pm\omega_{jk} \qquad \Rightarrow \qquad 2\frac{\partial_r c_i}{c_i} = \mp\varepsilon_{jki}\left(c_j^2 + c_k^2 - c_i^2\right).$$

$$\therefore \qquad \partial_r\left(\ln c_i^2\right) = \mp\varepsilon_{jki}\left(c_j^2 + c_k^2 - c_i^2\right) \tag{3.4.1.5}$$

One may suppose that we must parameterize the LHS to match the linear form of the RHS in the equation above. From (3.4.1.5), we can see that c_i^2 must be parameterized such that

$$\partial_r\left(\ln c_j^2 + \ln c_k^2\right) = \mp 2c_i^2 \quad \equiv \quad 2\partial_r\left(\ln\Omega_i\right) \qquad \Rightarrow \qquad \mp c_i^2 \equiv \partial_r\left(\ln\Omega_i\right), \tag{3.4.1.6}$$

which being applied back into the RHS of (3.4.1.5) leads us to:

$$\ln c_i^2 = \ln\Omega_j + \ln\Omega_k - \ln\Omega_i = \ln\left(\frac{\Omega_j\Omega_k}{\Omega_i}\right),$$

$$\therefore \qquad (c_i)^2 = \frac{\Omega_j\Omega_k}{\Omega_i} \qquad \Rightarrow \qquad \Omega_i = c_jc_k. \tag{3.4.1.7}$$

which enable us to decouple the individual parameters into their own equations turning into simpler expressions. Applying (3.4.1.7) to (3.4.1.6) allows us to write:

$$\frac{\dot\Omega_i}{\Omega_i} = \mp\frac{\Omega_j\Omega_k}{\Omega_i} \qquad \Rightarrow \qquad \boxed{\dot\Omega_i = \mp\Omega_j\Omega_k} \tag{3.4.1.8}$$

where throughout derivative (denoted by dot) is taken with respect to r.

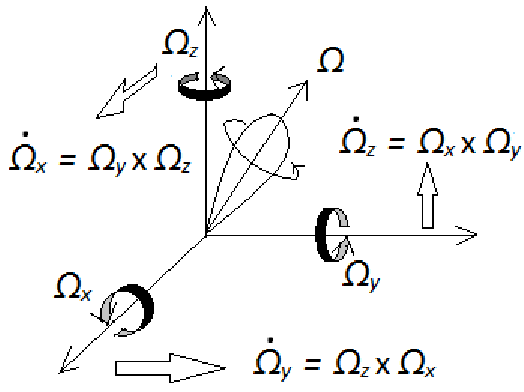

Figure 3.1: The Eüler Top mechanism

Curvature wise self-duality - Classical Darboux-Halphen system

Since we have already covered connection-wise self duality, let us explore a stronger version known as curvature-wise self-duality. This emphasizes and expands upon the property of self-duality, generalizing it beyond connection 1-forms. This means that curvature-wise self-duality does not invalidate, rather implies connection-wise self-duality [118], and hence part of the dynamical system derived from this should have the same form as the Lagrange system.

The Cartan-structure equation for Riemann curvature is

$$R_{ij} = d\omega_{ij} + \omega_{im} \wedge \omega^m{}_j \tag{3.4.1.9}$$

The (anti-) self-duality of curvature demands that

$$R_{0i} = \pm\frac{1}{2}\varepsilon_{0i}{}^{jk} R_{jk} = \pm R_{jk} \tag{3.4.1.10}$$

Now, for the LHS and RHS of (3.4.1.10), we have for $i \neq j \neq k \neq 0$

$$\text{LHS} \qquad R_{0i} = d\omega_{0i} + \omega_{0j} \wedge \omega^j{}_i + \omega_{0k} \wedge \omega^k{}_i \tag{3.4.1.11}$$

$$\begin{aligned} \text{RHS} \qquad R_{jk} &= d\omega_{jk} + \omega_{j0} \wedge \omega^0{}_k + \omega_{ji} \wedge \omega^i{}_k, \\ &= d\omega_{jk} - \omega_{0j} \wedge \omega^0{}_k - \omega_{ji} \wedge \omega_{ki} \end{aligned} \tag{3.4.1.12}$$

Using the (anti-) self-duality of the connection forms, as employed in the previous subsection,

$$\omega_{ij} = \pm\frac{1}{2}\varepsilon_{ij}{}^{k0}\omega_{k0} = \mp\omega_{k0} \qquad \Rightarrow \qquad \omega_{ji} \pm \omega_{0k} = 0 \tag{3.4.1.13}$$

we shall be able to eliminate some of the later terms of (3.4.1.11) and (3.4.1.12) in (3.4.1.10), since

$$\begin{aligned} R_{0i} = \pm R_{jk} \qquad &\Rightarrow \qquad R_{0i} \mp R_{jk} = 0, \\ \Rightarrow \qquad d\omega_{0i} \mp d\omega_{jk} + \omega_{0j} \wedge &\underbrace{\left(\omega^j{}_i \pm \omega^0{}_k\right)}_{0} + \underbrace{\left(\omega_{0k} \pm \omega_{ji}\right)}_{0} \wedge \omega_{ki} = 0. \end{aligned}$$

$$d\omega_{0i} = \pm d\omega_{jk} \tag{3.4.1.14}$$

This leaves us with the equation shown below and its solution that follows, adapted from before applying (3.4.1.2) and (3.4.1.4) into (3.4.1.14), remembering that $c_0 = c_i c_j c_k$.

$$\partial_r \left(\frac{\partial_r c_i}{c_0} \right) dr \wedge \sigma^i + \frac{\partial_r c_i}{c_0} d\sigma^i = \mp \partial_r \left(\frac{c_j^2 + c_k^2 - c_i^2}{2 c_j c_k} \right) dr \wedge \sigma^i \mp \frac{c_j^2 + c_k^2 - c_i^2}{2 c_j c_k} \, d\sigma^i,$$

$$\Rightarrow \qquad \partial_r \left(\frac{\partial_r c_i}{c_0} \right) = \mp \partial_r \left(\frac{c_j^2 + c_k^2 - c_i^2}{2 c_j c_k} \right) \qquad \Rightarrow \qquad \frac{\partial_r c_i}{c_0} = \mp \varepsilon_{jki} \frac{c_j^2 + c_k^2 - c_i^2}{2 c_j c_k} + \lambda_{jk},$$

$$\Rightarrow \qquad 2\partial_r \left(\ln c_i \right) = \mp \left(c_j^2 + c_k^2 - c_i^2 \right) + 2\lambda_{jk} c_j c_k, \tag{3.4.1.15}$$

where $\lambda_{jk} = \lambda_{kj}$. Thus, starting from (3.4.1.15) as before, we can parameterize as follows:

$$\partial_r \left(\ln c_j + \ln c_k \right) = \mp c_i^2 + c_i \left(\lambda_{ij} c_j + \lambda_{ik} c_k \right) \quad \equiv \quad \partial_r \left(\ln \Omega_i \right). \tag{3.4.1.16}$$

Applying the parametrization (3.4.1.16) back into (3.4.1.15), we get (3.4.1.7) all over again:

$$\partial_r \left[\ln (c_i)^2 \right] = \underbrace{\left[\mp c_j^2 + c_j \left(\lambda_{jk} c_k + \lambda_{ji} c_i \right) \right]}_{\partial_r (\ln \Omega_j)} + \underbrace{\left[\mp c_k^2 + c_k \left(\lambda_{ki} c_i + \lambda_{kj} c_j \right) \right]}_{\partial_r (\ln \Omega_k)} - \underbrace{\left[\mp c_i^2 + c_i \left(\lambda_{ij} c_j + \lambda_{ik} c_k \right) \right]}_{\partial_r (\ln \Omega_i)},$$

$$(c_i)^2 = \frac{\Omega_j \Omega_k}{\Omega_i} \qquad \Rightarrow \qquad \Omega_i = c_j c_k.$$

Applying this parametrization to (3.4.1.16) will now give us

$$\partial_r \left(\ln \Omega_i \right) = \mp \frac{\Omega_j \Omega_k}{\Omega_i} + \lambda_{ij} \Omega_k + \lambda_{ik} \Omega_j,$$

$$\Rightarrow \qquad \dot{\Omega}_k = \mp \Omega_i \Omega_j + \lambda_{jk} \Omega_k \Omega_i + \lambda_{ik} \Omega_k \Omega_j \tag{3.4.1.17}$$

where setting $\lambda_{ij} = -1 \quad \forall \quad i, j$ in (3.4.1.17) for anti-self-duality proceeds to give us the classical Darboux-Halphen system

$$\therefore \qquad \boxed{\dot{\Omega}_k = \Omega_i \Omega_j - \Omega_k \left(\Omega_i + \Omega_j \right)} \tag{3.4.1.18}$$

Thus, we can see that the curvature-wise self-duality extends upon the characteristic system of the connection-wise self-duality, making the Darboux-Halphen system a suitable candidate for further development beyond the Lagrange system. Clearly, the first term has included the dynamical aspect of the Lagrange system, as the property of self-duality of the connection 1-forms being preserved, aside from an additive constant involved and was extended to their exterior derivatives. Needless to say, connection-wise self-duality must precede curvature-wise self-duality, and the latter is not possible without ensuring the former.

Self-dual curvature components

So far, we have managed to study a great deal about the Bianchi-IX geometry, without confronting the work of extracting the curvature components. Here, we will proceed to do exactly that, using the imposed (anti-) self duality properties at our disposal to make our job easier. But first, we shall prove and later in this case confirm that all curvature-wise self-dual manifolds are Ricci-flat.

We recall from [117] that the Riemann curvature tensor for (anti-)self-dual metrics on the vierbein space can be written as:

$$R_{abcd} = \mathcal{G}_{ij}(\vec{x})\eta^{(\pm)i}_{ab}\eta^{(\pm)j}_{cd} \qquad i,j=1,2,3; \quad a,b,c,d=0,1,2,3 \tag{3.4.1.19}$$

This means the Ricci tensor for Euclidean vierbein space is given by

$$\mathbb{R}_{ac} = \delta^{bd}R_{abcd} = \delta^{bd}\mathcal{G}_{ij}(\vec{x})\eta^i_{ab}\eta^j_{cd} = \mathcal{G}_{ij}(\vec{x})\delta^{ij}\delta_{ac} = \Big(\mathrm{Tr}\left[\mathcal{G}(\vec{x})\right]\Big)\delta_{ac} \tag{3.4.1.20}$$

Clearly, the above result implies that the Ricci tensor has only diagonal elements, which allows us to demonstrate that

$$\mathbb{R}_{aa} = R_{abab} + R_{acac} + R_{adad} \xrightarrow{\text{self-duality}} \pm\left(R_{abcd} + R_{acdb} + R_{adbc}\right) \xrightarrow{\text{Bianchi identity}} 0,$$

$$\therefore \qquad \boxed{\mathbb{R}_{aa} = 0} \qquad \Rightarrow \qquad \boxed{\mathrm{Tr}\left[\mathcal{G}(\vec{x})\right] = 0} \tag{3.4.1.21}$$

Showing that curvature-wise self-dual manifolds are undoubtedly Ricci-flat.

$$\boxed{\boxed{\text{Self-duality} \quad \Longrightarrow \quad \text{Ricci-flatness}}}.$$

Returning to the original co-ordinates, we have:

$$\mathbb{R}_{ac} = \left(\mathcal{G}_{ij}(\vec{x})\delta^{ij}\right)\left(\delta_{ac}e^a_\mu e^c_\nu\right) = \mathrm{Tr}\left[\mathcal{G}(\vec{x})\right]g_{\mu\nu}(\vec{x}) \tag{3.4.1.22}$$

But, for a more thorough analysis, it would be better to directly obtain all the curvature components for detailed examination. This can be easily done as we already have the general formula for all the connection components. The results are made easier by keeping the self-duality of the connection forms (3.4.1.13) in mind.

$$\begin{aligned} R_{0i} &= d\omega_{0i} + \omega_{0j}\wedge\omega_{ji} + \omega_{0k}\wedge\omega_{ki} \\ &= d\omega_{0i} + \omega_{0j}\wedge\omega_{0k} - \omega_{0k}\wedge\omega_{0j} = d\omega_{0i} + 2\omega_{0j}\wedge\omega_{0k} \end{aligned}$$

$$\therefore \qquad R_{0i} = \underbrace{\frac{1}{c_0c_i}\left(\frac{c_i'}{c_0}\right)'}_{R_{0i0i}} e^0\wedge e^i - \underbrace{\frac{1}{c_jc_k}\left[\frac{c_i'}{c_0} - 2\frac{\left(c_j'\right)\left(c_k'\right)}{c_0^2}\right]}_{-R_{0ijk}} e^j\wedge e^k \tag{3.4.1.23}$$

Now curvature wise anti-self-duality means

$$R_{0i0i} = -R_{0ijk} = -R_{jk0i} = R_{jkjk} \tag{3.4.1.24}$$

Demanding curvature wise anti-self-duality gives us the differential equation

$$\frac{1}{c_0c_i}\left(\frac{c_i'}{c_0}\right)' = \frac{1}{c_jc_k}\left[\frac{c_i'}{c_0} - 2\frac{\left(c_j'\right)\left(c_k'\right)}{c_0^2}\right] \tag{3.4.1.25}$$

Since we have connection wise anti-self-duality rule (3.4.1.5), we can say

$$R_{0i} = \frac{\varepsilon_{ijk}}{c_0c_i}\left(\frac{c_j^2+c_k^2-c_i^2}{2c_jc_k}\right)' e^0\wedge e^i - \frac{\varepsilon_{ijk}}{c_jc_k}\left[\frac{c_j^2+c_k^2-c_i^2}{2c_jc_k} - 2\left(\frac{c_k^2+c_i^2-c_j^2}{2c_kc_i}\right)\left(\frac{c_i^2+c_j^2-c_k^2}{2c_ic_j}\right)\right] e^j\wedge e^k$$

$$\therefore \quad R_{0i} = \underbrace{\frac{\varepsilon_{ijk}}{c_0c_i}\left(\frac{c_j^2+c_k^2-c_i^2}{2c_jc_k}\right)'}_{R_{0i0i}} e^0\wedge e^i - \underbrace{\varepsilon_{ijk}\frac{c_i^2\left(c_j^2+c_k^2-c_i^2\right) - c_i^4 + \left(c_j^2-c_k^2\right)^2}{2c_0^2}}_{-R_{0ijk}} e^j\wedge e^k \tag{3.4.1.26}$$

Due to curvature wise anti-self-duality being considered, we should have:

$$R_{0i0i} = -R_{0ijk} = \varepsilon_{ijk} \frac{\left(c_i^2 + c_j^2 + c_k^2\right)\left(c_j^2 + c_k^2\right) - 2c_i^4 - 4c_j^2 c_k^2}{2c_0^2} \tag{3.4.1.27}$$

Thus, we can say that the curvature 2-form is given by

$$\boxed{R_{ab} = \sum_{i=1}^{3} \varepsilon_{ijk} \frac{\left(c_i^2 + c_j^2 + c_k^2\right)\left(c_j^2 + c_k^2\right) - 2c_i^4 - 4c_j^2 c_k^2}{2c_0^2}\ \bar{\eta}^i_{ab}\bar{\eta}^i_{cd} e^c \wedge e^d} \tag{3.4.1.28}$$

which on comparison with (3.4.1.19) tells us that

$$\mathcal{G}_{il}(\vec{x}) = \varepsilon_{ijk} \frac{\left(c_i^2 + c_j^2 + c_k^2\right)\left(c_j^2 + c_k^2\right) - 2c_i^4 - 4c_j^2 c_k^2}{2c_0^2} \delta_{il} \tag{3.4.1.29}$$

$$\mathrm{Tr}\left[\mathcal{G}\right] = \frac{2\left(c_i^2 + c_j^2 + c_k^2\right)^2 - 2\left(c_i^2 + c_j^2 + c_k^2\right)^2}{2c_0^2} = 0 \tag{3.4.1.30}$$

Thus, the Ricci tensor, and consequently scalar on vierbein space is given by

$$\boxed{\mathbb{R}_{ab} = 0, \qquad \mathbb{R} = 0} \tag{3.4.1.31}$$

confirming what was already proven previously.

Non-existence of a metric for the generalized system

Naturally, one can suspect that the classical Darboux-Halphen system and consequently the Bianchi-IX metric is the result of setting $\tau_i = 0$ in (3.3.1.1). For the classical system, we should have the metric co-efficients given by the diagonal matrix

$$h_{class} = \begin{pmatrix} \Omega_1\Omega_2\Omega_3 & 0 & 0 & 0 \\ 0 & \dfrac{\Omega_2\Omega_3}{\Omega_1} & 0 & 0 \\ 0 & 0 & \dfrac{\Omega_3\Omega_1}{\Omega_2} & 0 \\ 0 & 0 & 0 & \dfrac{\Omega_1\Omega_2}{\Omega_3} \end{pmatrix} \tag{3.4.1.32}$$

Now, we notice that the matrix describing the metric h can be given by

$$h_{class} = M_{class}^{-1}\,\mathrm{Adj}\left(M_{class}\right) \qquad \text{where} \qquad M_{class} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \Omega_1 & 0 & 0 \\ 0 & 0 & \Omega_2 & 0 \\ 0 & 0 & 0 & \Omega_3 \end{pmatrix} \tag{3.4.1.33}$$

where M_{class} is the matrix that produces the classical Darboux-Halphen system. With this in mind, we see that the generalized Darboux-Halphen system (3.3.1.1) seems to arise from a matrix M_{gen} given as

$$M_{gen} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \Omega_1 & \tau_3 & -\tau_2 \\ 0 & -\tau_3 & \Omega_2 & \tau_1 \\ 0 & \tau_2 & -\tau_1 & \Omega_3 \end{pmatrix} \tag{3.4.1.34}$$

However, there are not always a vierbeins or metric counterparts for gauge fields, as there are for curvature and connection components. This shall be elaborated further as follows.

The torsion-free form of the 1st Cartan structure equation is

$$de^i = -\omega^i{}_j \wedge e^j \tag{3.4.1.35}$$

Further examination reveals that

$$\partial_\mu e^i{}_\nu \, dx^\mu \wedge dx^\nu = -\omega^i_{\mu j} e^j{}_\nu \, dx^\mu \wedge dx^\nu,$$

$$\therefore \qquad E_j{}^\nu \partial_\mu e^i{}_\nu = -\omega^i_{\mu j} \tag{3.4.1.36}$$

Recalling that the spin connections can be expanded as shown below, we can say that

$$\omega_{ij} = \eta^{(+)a}_{ij} A^{(+)a} + \eta^{(-)a}_{ij} A^{(-)a} \qquad\qquad A^{(\pm)a} = A^{(\pm)a}_\mu dx^\mu,$$

$$\therefore \qquad E_j{}^\nu \partial_\mu e^i{}_\nu = -\left[\eta^{(+)a}_{ij} A^{(+)a}_\mu + \eta^{(-)a}_{ij} A^{(-)a}_\mu\right].$$

Now, we can obtain the individual $\mathrm{SU}(2)_\pm$ gauge potential function components $A^{(\pm)}_\mu$ in terms of the vierbein components as follows

$$A^{(\pm)a}_\mu = -\frac{1}{4}\eta^{(\pm)a} E_j{}^\nu \partial_\mu e^i{}_\nu \tag{3.4.1.37}$$

Thus, if we start with a metric or equivalently the vierbeins, we should be able to get the spin-connections and hence gauge fields, and from there go backwards, however, the opposite is not always possible.

Since the generalized Darboux-Halphen system is primarily a product of the reduction of the SDYM gauge fields, it may not always be possible to find a metric or its vierbeins that are related to it. The classical Darboux-Halphen system is a special case where $\tau_i = 0 \quad \forall \quad i$, for which we have the self-dual Bianchi-IX metric (gravitational instanton).

3.4.2 Aspects of Flow equations

Geometric flows describe the evolution of a metric on a Riemannian manifold along the path parameter, under a general non-linear equation, given a symmetric tensor S_{ij} [119, 120]. Usually, a system that exhibits geometric flows satisfies the equation

$$\frac{dg_{ij}}{d\tau} = S_{ij} \tag{3.4.2.1}$$

where S_{ij} is symmetric. Some systems exhibit a particular category of such flows known as Ricci flow for which $S_{ij} = -R_{ij}$. Such systems that describe Ricci flows do not preserve volume elements, which are described by the equation:

$$\frac{dg_{ij}}{d\tau} = -R_{ij} \tag{3.4.2.2}$$

The Ricci flow equation introduced by Richard Hamilton in 1982 was a primary tool in Grigory Perelman's proof of Thurston's geometrization conjecture, Poincare conjecture being a special case of that. Ricci flow exhibits many similarities with the heat equation: it gives manifolds more uniform geometry and smooths out irregularities and has proven to be a very useful tool in understanding the topology of arbitrary Riemannian manifolds.

Now, we have already shown that Darboux-Halphen systems are Ricci-flat which means that it is a fixed point of the Ricci flow, usually exhibited by gravitational instantons which are extremal points of the Euclidean Einstein-Hilbert action [121]. Looking at the Darboux-Halphen equations, we can see that for the Bianchi-IX metric

$$\frac{d\left(c_i^2\right)}{d\tau} = c_i^2\left(c_j^2 + c_k^2 - c_i^2 - 2c_jc_k\right) = c_i^2\left[\left(c_j - c_k\right)^2 - c_i^2\right],$$

$$\frac{d\left(c_0^2\right)}{d\tau} = \frac{d}{d\tau}\left(c_1c_2c_3\right)^2 = c_0^2\left\{c_i^2 + c_j^2 + c_k^2 - 2\left(c_ic_j + c_jc_k + c_kc_i\right)\right\}.$$

Thus, we have the following equations:

$$\boxed{\begin{aligned} \frac{d\left(c_0^2\right)}{d\tau} &= c_0^2(\vec{x})\left[c_i^2 + c_j^2 + c_k^2 - 2\left(c_ic_j + c_jc_k + c_kc_i\right)\right] \\ \frac{d\left(c_i^2\right)}{d\tau} &= c_i^2(\vec{x})\left[\left(c_j - c_k\right)^2 - c_i^2\right] \end{aligned}} \tag{3.4.2.3}$$

Now if we set $c_0 = 1$ for the co-ordinate rescaling $dt = c_0(\vec{x})\ dr$, then we should get

$$\boxed{\begin{aligned} c_i^2 + c_j^2 + c_k^2 &= 2\left(c_ic_j + c_jc_k + c_kc_i\right) \\ 2\frac{dc_i}{d\tau} &= \frac{1}{c_jc_k}\left[\left(c_j - c_k\right)^2 - c_i^2\right] \end{aligned}} \tag{3.4.2.4}$$

which matches and re-confirms the results obtained in [120] where the Ricci tensor for such Bianchi-IX geometry is given by the RHS of the above equation, showing that it does exhibit Ricci flow. For a more general Darboux-Halphen system, the result would be of the form:

$$\frac{d\left(c_i^2\right)}{d\tau} = c_i^2\left[c_j^2 + c_k^2 - c_i^2 - 2\left(\beta_{ij}\ c_ic_j + \beta_{jk}\ c_jc_k + \beta_{ki}\ c_kc_i\right)\right].$$

where $\quad 2\beta_{ij} = \lambda_{jk} - \lambda_{ik}, \quad 2\beta_{jk} = \lambda_{ji} + \lambda_{ki}, \quad 2\beta_{ki} = \lambda_{kj} - \lambda_{ij}$

and $$\frac{d\left(c_0^2\right)}{d\tau} = \frac{d}{d\tau}\left(c_1c_2c_3\right)^2 = c_0^2(\vec{x})\left[c_i^2 + c_j^2 + c_k^2 - 2\left(\alpha_{ij}\ c_ic_j + \alpha_{jk}\ c_jc_k + \alpha_{ki}\ c_kc_i\right)\right].$$

where $2\alpha_{ij} = \lambda_{ik} + \lambda_{jk}, \quad 2\alpha_{jk} = \lambda_{ji} + \lambda_{ki}, \quad 2\alpha_{ki} = \lambda_{kj} + \lambda_{ij}$

$$\boxed{\begin{aligned} \frac{d\left(c_0^2\right)}{d\tau} &= c_0^2(\vec{x})\left[c_i^2 + c_j^2 + c_k^2 - 2\left(\alpha_{ij}\ c_ic_j + \alpha_{jk}\ c_jc_k + \alpha_{ki}\ c_kc_i\right)\right] \\ \frac{d\left(c_i^2\right)}{d\tau} &= c_i^2(\vec{x})\left[c_j^2 + c_k^2 - c_i^2 - 2\left(\beta_{ij}\ c_ic_j + \beta_{jk}\ c_jc_k + \beta_{ki}\ c_kc_i\right)\right] \end{aligned}} \tag{3.4.2.5}$$

Thus, Ricci flow is exhibited and implies a self-dual Bianchi-IX metric, otherwise known to be the Darboux-Halphen system describing the evolution of $SU(2)$ 3D geometries.

3.4.3 Other related systems

The Darboux-Halphen system has analogues and equivalents in various forms of quadratic and non-linear differential equations. In this subsection, we will describe them in detail.

Ramamani to Darboux-Halphen

The Ramamani system [122, 123] is described by the following differential equations

$$q\frac{d\mathcal{P}}{dq} = \frac{\mathcal{P}^2 - \mathcal{Q}}{4}, \qquad q\frac{d\widetilde{\mathcal{P}}}{dq} = \frac{\widetilde{\mathcal{P}}\mathcal{P} - \mathcal{Q}}{2}, \qquad q\frac{d\mathcal{Q}}{dq} = \mathcal{P}\mathcal{Q} - \widetilde{\mathcal{P}}\mathcal{Q}. \tag{3.4.3.1}$$

In a recent paper Ablowitz et al. [124] showed that Ramamani's system of differential equations is equivalent to a third order scalar nonlinear ODE found by Bureau [125], whose solutions are given implicitly by a Schwarzian triangle function. Under a suitable variable transformation, the Ramamani system produces the classical Darboux-Halphen system.

The Ramamani system (3.4.3.1) for $q = \dfrac{1}{2i\pi}$, is described by the equations

$$\dot{\mathcal{P}} = \frac{i\pi}{2}\left(\mathcal{P}^2 - \mathcal{Q}\right), \qquad \dot{\widetilde{\mathcal{P}}} = i\pi\left(\mathcal{P}\widetilde{\mathcal{P}} - \mathcal{Q}\right), \qquad \dot{\mathcal{Q}} = 2i\pi\left(\mathcal{P} - \widetilde{\mathcal{P}}\right)\mathcal{Q}. \tag{3.4.3.2}$$

We convert to Darboux-Halphen variables $\left(\mathcal{P}, \widetilde{\mathcal{P}}, \mathcal{Q}\right) \to (X, Y, Z)$ [126] as follows

$$\mathcal{P} = \frac{2}{i\pi}X, \qquad \widetilde{\mathcal{P}} = \frac{1}{i\pi}\left(2X - Y - Z\right), \qquad \mathcal{Q} = \frac{4}{\pi^2}\left(Z - X\right)\left(X - Y\right). \tag{3.4.3.3}$$

Naturally, if we apply the above transformation to the Ramamani equations (3.4.3.2), we shall get the classical Darboux-Halphen system equations.

$$\begin{aligned} \dot{\mathcal{P}} = \frac{2}{i\pi}\dot{X} &= \frac{i\pi}{2}\left\{-\frac{4}{\pi^2}X^2 - \frac{4}{\pi^2}\left(XY + XZ - YZ - X^2\right)\right\}, \\ \Rightarrow \qquad -\frac{4}{\pi^2}\dot{X} &= -\frac{4}{\pi^2}\left(XY + XZ - YZ\right) \end{aligned}$$

and hence, we get one Darboux-Halphen equation in the form

$$\dot{X} = X\left(Y + Z\right) - YZ \tag{3.4.3.4}$$

For the others, the process is more elaborate although quite straightforward to show.

$$\begin{aligned} \dot{\widetilde{\mathcal{P}}} = \frac{1}{i\pi}\left(2\dot{X} - \dot{Y} - \dot{Z}\right) &= i\pi\left\{-\frac{4}{\pi^2}X^2 + \frac{2}{\pi^2}X\left(Y + Z\right) - \frac{4}{\pi^2}\left(XY + XZ - YZ - X^2\right)\right\} \\ \Rightarrow \qquad -\frac{1}{\pi^2}\left(2\dot{X} - \dot{Y} - \dot{Z}\right) &= \frac{2}{\pi^2}\left(\dot{X} + YZ\right) - \frac{4}{\pi^2}\dot{X} \end{aligned}$$

$$\Rightarrow \qquad \dot{Y} + \dot{Z} = 2YZ \tag{3.4.3.5}$$

$$\begin{aligned} \dot{\mathcal{Q}} = \frac{4}{\pi^2}\left[\left(\dot{Z} - \dot{X}\right)\left(X - Y\right) + \left(Z - X\right)\left(\dot{X} - \dot{Y}\right)\right] &= \frac{8}{\pi^2}\left(Y + Z\right)\left(Z - X\right)\left(X - Y\right), \\ \Rightarrow \qquad \left(\dot{Z} - Z^2\right)\left(X - Y\right) + \left(Y^2 - \dot{Y}\right)\left(Z - X\right) &= 0. \end{aligned}$$

Using (3.4.3.5), we have another DH equation

$$\dot{Y} = Y\left(Z + X\right) - ZX \tag{3.4.3.6}$$

Naturally, using (3.4.3.5) again, we should get the final equation

$$\dot{Z} = Z\left(X + Y\right) - XY \tag{3.4.3.7}$$

Consequently, we have three sets of equations for the classical Darboux-Halphen system

$$\boxed{\begin{aligned}\dot{X} &= X\left(Y+Z\right)-YZ\\ \dot{Y} &= Y\left(Z+X\right)-ZX\\ \dot{Z} &= Z\left(X+Y\right)-XY\end{aligned}}$$

This is henceforth, another system related to the Ramanujan equations [127, 128] via the focal point of classical DH systems they converge to. We must note that, the Ramamani system gives rise to self dual (and not anti-self dual) Darboux-Halphen equations. Inverting the sign of the Halphen variables gives the familiar anti-self-dual system.

The Chazy equation

Now, we shall see how the solution of the Darboux-Halphen system satisfies the Chazy equation [129, 130]. Let us take the previous result for anti-self-duality and $\lambda_{ij} = -1$, and write it for all values of i, j, k

$$\begin{aligned}\dot{\Omega}_1 &= \Omega_2\Omega_3 - \Omega_1\left(\Omega_2+\Omega_3\right)\\ \dot{\Omega}_2 &= \Omega_3\Omega_1 - \Omega_2\left(\Omega_3+\Omega_1\right)\\ \dot{\Omega}_3 &= \Omega_1\Omega_2 - \Omega_3\left(\Omega_1+\Omega_2\right)\end{aligned}$$

Adding up will give

$$\dot{\Omega}_1+\dot{\Omega}_2+\dot{\Omega}_3 = -\left(\Omega_1\Omega_3+\Omega_2\Omega_1+\Omega_3\Omega_2\right).$$

If we define $y = -2\left(\Omega_1+\Omega_2+\Omega_3\right)$, we will have

$$\dot{y} = 2\left(\Omega_1\Omega_3+\Omega_2\Omega_1+\Omega_3\Omega_2\right), \qquad \ddot{y} = -12\ \Omega_1\Omega_2\Omega_3.$$

Thus, the third order derivative will be

$$\begin{aligned}\dddot{y} &= -12\Big[\left\{\Omega_2\Omega_3-\Omega_1\left(\Omega_2+\Omega_3\right)\right\}\Omega_2\Omega_3+\Omega_1\left\{\Omega_3\Omega_1-\Omega_2\left(\Omega_3+\Omega_1\right)\right\}\Omega_3\\ &\qquad +\Omega_1\Omega_2\left\{\Omega_1\Omega_2-\Omega_3\left(\Omega_1+\Omega_2\right)\right\}\Big]\\ &= 48\ \Omega_1\Omega_2\Omega_3\left(\Omega_1+\Omega_2+\Omega_3\right)-12\ \left(\Omega_2\Omega_3+\Omega_3\Omega_1+\Omega_1\Omega_2\right)^2 \ = \ 2y\ddot{y}-3\dot{y}^2\end{aligned}$$

Thus, we obtain the Chazy equation [129]

$$\boxed{\frac{d^3y}{dt^3} = 2y\frac{d^2y}{dt^2}-3\left(\frac{dy}{dt}\right)^2} \tag{3.4.3.8}$$

The Ramanujan equation

In case of the classical Chazy system, the Ramanujan equations [127, 128] are given by

$$\dot{P} = \frac{i\pi}{6}\left(P^2-Q\right), \qquad \dot{Q} = \frac{2i\pi}{3}\left(PQ-R\right), \qquad \dot{R} = i\pi\left(PR-Q^2\right). \tag{3.4.3.9}$$

To understand how they are related to the Chazy equation, we shall examine what they imply systematically. From the first equation of (3.4.3.9), we find that

$$Q = P^2-\frac{6}{i\pi}\dot{P} \qquad \Rightarrow \qquad Q = Q(P,\dot{P}) \tag{3.4.3.10}$$

Applying (3.4.3.10) to the second eq. of (3.4.3.9), we get

$$\dot{Q} = \dot{Q}(P, \dot{P}, \ddot{P}) = 2\left(P\dot{P} - \frac{3}{i\pi}\ddot{P}\right),$$

$$\Rightarrow \quad R = R(P, \dot{P}, \ddot{P}) = PQ - \frac{3}{2i\pi}\dot{Q} = P^3 - \frac{9}{i\pi}P\dot{P} - \frac{9}{\pi^2}\ddot{P} \tag{3.4.3.11}$$

Finally, using result (3.4.3.11) in the last equation of (3.4.3.9), we will get

$$\dot{R} = i\pi\left(PR - Q^2\right) = 3P^2\dot{P} - \frac{9}{i\pi}\left(\dot{P}^2 + P\ddot{P}\right) - \frac{9}{\pi^2}\dddot{P},$$

$$\therefore \qquad \dddot{P} + i\pi\left(3\dot{P}^2 - 2P\ddot{P}\right) = 0.$$

However, we are not there yet. The final step requires us to take advantage of the non-linearity of this equation and write $P = \frac{y}{i\pi}$ to arrive at

$$\dddot{y} = 2y\ddot{y} - 3\dot{y}^2 \tag{3.4.3.12}$$

This final result is the classical Chazy equation (3.4.3.8) we are familiar with.

Generalized Chazy System

If we start instead with the generalized Darboux-Halphen equations and set the co-efficients as $\alpha_1 = \alpha_2 = \alpha_3 = \frac{2}{n}$, we will get the corresponding generalized Chazy equation [131].

$$\boxed{\frac{d^3y}{dt^3} - 2y\frac{d^2y}{dt^2} + 3\frac{dy}{dt}^2 = \frac{4}{36 - n^2}\left(6\frac{dy}{dt} - y^2\right)^2} \tag{3.4.3.13}$$

The set of transformations that leads the Ramanujan equations to the above generalized Chazy equation turn out to be:

$$\dot{P} = \frac{i\pi}{6}\left(P^2 - Q\right), \qquad \dot{Q} = \frac{2i\pi}{3}\left(PQ - R\right), \qquad \dot{R} = i\pi\left[PR - Q^2\left(1 - \frac{36}{36 - n^2}\right)\right] \tag{3.4.3.14}$$

The first two equations of (3.4.3.14) are the same as for (3.4.3.9), so the same steps will follow as with (3.4.3.10) and (3.4.3.11), but for the last step, we will have

$$\dddot{P} + i\pi\left(3\dot{P}^2 - 2P\ddot{P}\right) = \frac{4i\pi}{36 - n^2}\left(P^2 - \frac{6}{i\pi}\dot{P}\right)^2 \tag{3.4.3.15}$$

Applying the same variable redefinition $P = \frac{y}{i\pi}$ as before, we obtain

$$\frac{d^3y}{dt^3} - 2y\frac{d^2y}{dt^2} + 3\frac{dy}{dt}^2 = \frac{4}{36 - n^2}\left(6\frac{dy}{dt} - y^2\right)^2 \tag{3.4.3.16}$$

which is exactly the generalized Chazy equation described before.

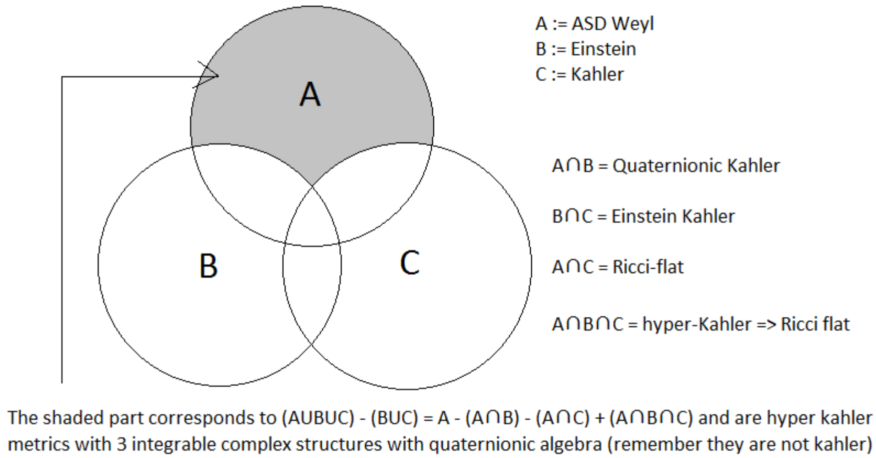

Figure 3.2: Venn-diagram of Quaternionic, Einstein Kähler, and Ricci-flat manifolds

3.4.4 Integrability of the Bianchi-IX

There are various impositions possible on a 4-dimensional Riemannian metric. It could be Kähler or Einstein or even have an anti-self-dual (ASD) Weyl tensor. The venn-diagram below depicts the various possibilities resulting to different field equations in 4-dimensions, where the intersection zones correspond to interesting conditions.
If (e^0, e^1, e^2, e^3) define the vierbeins on a Riemannian 4-manifold, the basis of self-dual 2-forms is given as

$$*\,\eta^i = \eta^i = \eta^i_{ab} e^a \wedge e^b : \begin{cases} \eta^1 = e^0 \wedge e^1 + e^2 \wedge e^3 \\ \eta^2 = e^0 \wedge e^2 + e^3 \wedge e^1 \\ \eta^3 = e^0 \wedge e^3 + e^1 \wedge e^2 \end{cases} \tag{3.4.4.1}$$

Similarly, the anti-self-dual 2-form basis is given by

$$*\,\bar{\eta}^i = -\bar{\eta}^i = \bar{\eta}^i_{ab} e^a \wedge e^b : \begin{cases} \bar{\eta}^1 = e^0 \wedge e^1 - e^2 \wedge e^3 \\ \bar{\eta}^2 = e^0 \wedge e^2 - e^3 \wedge e^1 \\ \bar{\eta}^3 = e^0 \wedge e^3 - e^1 \wedge e^2 \end{cases} \tag{3.4.4.2}$$

If ${\omega^i}_j$ are the self-dual parts spin connection 1-forms, then the first Cartan equation are

$$d\eta^i = {\omega^i}_j \wedge \eta^j \tag{3.4.4.3}$$

The curvature 2-form is given as usual by the 2nd Cartan structure equation (3.4.1.9). It is possible to expand the curvature in terms of η^i and $\bar{\eta}^i$ as

$$R_{ij} = {W_{ij}}^k \eta_k + {\Phi_{ij}}^k \bar{\eta}_k \tag{3.4.4.4}$$

where conditions imposed by various field equations determine the co-efficients W_{ijk} & Φ_{ijk}.

Conditions determining W_{ijk} & Φ_{ijk}

The 1st Bianchi identity ${R^i}_j \wedge \eta^j = 0$ implies $W_{ijj} = 0$, further implying that W_{ijk} has 6 independent components, out of which 5 correspond to the SD Weyl tensor and one to the totally anti-symmetric part corresponding to Ricci scalar. On the other hand, Φ_{ijk} has 9 components corresponding to trace-free Ricci tensor.

1. Iff $W_{ijk} = \Lambda \epsilon_{ijk}$, where Λ is a multiple of Ricci scalar, we are in Set A $\Rightarrow$ ASD Weyl.
2. Iff $W_{ijk} = \Lambda \epsilon_{ijk}$ and $\Phi_{ijk} = 0$, then we have ASD Einstein ($A \cap B$).
3. Iff $W_{ijk} = 0 = \Phi_{ijk}$, we are in ($A \cap B \cap C$) which is hyper-Kähler.

Returning to the Bianchi-IX metric (3.4.1.1) and using the parametrization (3.4.1.7), it can be written in terms of the basis $(\sigma^1, \sigma^2, \sigma^3)$ of left-invariant forms on $SU(2)$ as

$$ds^2 = [\Omega_1(r)\Omega_2(r)\Omega_3(r)]\, dr^2 + \frac{\Omega_2\Omega_3}{\Omega_1}\sigma_1^2 + \frac{\Omega_3\Omega_1}{\Omega_2}\sigma_2^2 + \frac{\Omega_1\Omega_2}{\Omega_3}\sigma_3^2 \tag{3.4.4.5}$$

where $\Omega_i, \forall i = 1, 2, 3$ are functions of r and σ^is satisfy Maurer Cartan equations. From this form of the metric, the vierbeins can be used to produce the SD 2-forms:

$$\eta^i = \Omega_j\Omega_k\ dr \wedge \sigma^i + \Omega_i\ \sigma^j \wedge \sigma^k \qquad i \neq j \neq k \tag{3.4.4.6}$$

Thus, the connection forms ω_{ij} can be written in terms of arbitrary functions $A_i(r), i = 1, 2, 3$ such that

$$\omega_{12} = \frac{A_3}{\Omega_3}\sigma^3 + (\text{cyclic permutations}) \tag{3.4.4.7}$$

All A_i components are obtained from the system below:

$$\dot{\Omega}_i = \Omega_j\Omega_k - \Omega_i\,(A_j + A_k) \qquad i \neq j \neq k = 1, 2, 3 \tag{3.4.4.8}$$

We will refer to this system as the first system for future reference.

With the help of Cartan calculus, one can find the curvature 2-forms in terms of derivatives of A_is. With the specific choice of field equations from restricting ourselves to a specific region of the diagram (indicated in the Venn diagram), we will obtain a second system of 1st order differential equations involving A_i.

If we choose regions outside the top circle, we typically get non-integrable equations. Dancer and Strachan [132] already showed this for Einstein Kähler ($B \cap C$), while Barrow [133] showed the same thing for Einstein (Set B). However, field equations belonging to the top circle A are integrable, as expected from the heuristic, yet concrete argument that self-duality implies integrability. according to Mason [134].

Imposing the vanishing of ASD Weyl tensor and the scalar curvature ω_{ij} results in the system of the equation widely known as Chazy system [108, 135]. This system has a long history, having been studied and solved in the 19th century by Brioschi [136].

$$\dot{A}_i = A_jA_k - A_i\,(A_j + A_k) \qquad i \neq j \neq k = 1, 2, 3 \tag{3.4.4.9}$$

We shall now list the following features of the first and second systems:

i) If all $A_i = 0$, then the connection of ASD 2-forms are clearly flat and the metric describes vacuum. This was found by Belinsky [137] and Eguchi-Hanson [138].

ii) If $\Omega_i = A_i, \forall\ i$, then the first and second systems of equations are identical, which is precisely the Atiyah and Hitchin's [112] case.

iii) If one insists all A_is to be constant in r, then without any loss of generality, if two of them necessarily vanish, then the remaining $A_i \neq 0$ can reduce the first system to a special case of Painlevé-III [139]. Also, the form η^i is covariant constant in this case so that one arrives at the Pederson-Poon scalar flat Kähler metric [140].

iv) There exists a significant conserved quantity

$$Q = \frac{\Omega_1^2}{(A_1 - A_2)(A_1 - A_3)} + \frac{\Omega_2^2}{(A_2 - A_1)(A_2 - A_3)} + \frac{\Omega_3^2}{(A_3 - A_1)(A_3 - A_2)} \tag{3.4.4.10}$$

There is a covariance under fractional linear transformations in r [139], which means that the solutions of the second system with $A_1 = A_2 \neq 0$ is conformally related to the Pederson-Poon case [140].

Now, for a general solution of the second system, we introduce a new dependent variable x as per Brioschi [136]

$$x = \frac{A_1 - A_2}{A_3 - A_2} \tag{3.4.4.11}$$

It is now straightforward to show that (3.4.4.9) reduces to the 3rd order ODE for x

$$\dddot{x} = \frac{3}{2}\frac{\ddot{x}^2}{\dot{x}} - \frac{1}{2}(\dot{x})^3 \left(\frac{1}{x^2} + \frac{1}{x(x-1)} + \frac{1}{(x-1)^2} \right) \tag{3.4.4.12}$$

A remarkable fact is that this ODE is satisfied by the reciprocal of the elliptic modular function. Now this elliptic modular function has a natural boundary in the r-plane, so the A_i and hence Ω_i have a natural boundary in the r-plane and the location of the boundary depends on the constants of integration. This implies the self-duality equations are not always equivalent to Painlevé property, and thus integrable.

Now we introduce new dependent variables $\rho_i, \forall i = 1, 2, 3$ according to

$$\Omega_1 = \rho_1 \frac{\dot{x}}{\sqrt{x(1-x)}} \qquad \Omega_2 = \rho_2 \frac{\dot{x}}{x\sqrt{(1-x)}} \qquad \Omega_3 = \rho_3 \frac{\dot{x}}{\sqrt{x}(1-x)} \tag{3.4.4.13}$$

and switch independent variable from r to x (ie. $\dot{x} \equiv \frac{dx}{dr}$), so the first system becomes

$$\frac{d\rho_1}{dx} = \frac{\rho_2\rho_3}{x(1-x)} \qquad \frac{d\rho_2}{dx} = \frac{\rho_3\rho_1}{x} \qquad \frac{d\rho_3}{dx} = \frac{\rho_1\rho_2}{(1-x)} \tag{3.4.4.14}$$

This system is known to reduce to Painlevé VI with the first integral

$$\gamma = \frac{1}{2}\left(\rho_2^2 + \rho_3^2 - \rho_1^2\right) = const \tag{3.4.4.15}$$

which is in fact the conserved quantity (3.4.4.10), with the new metric being

$$ds^2 = \frac{\rho_1\rho_2\rho_3}{x(1-x)}\dot{x}\left[\frac{dx^2}{x(1-x)} + \frac{(\sigma^1)^2}{\rho_1^2} + \frac{(1-x)(\sigma^2)^2}{\rho_2^2} + \frac{x(\sigma^3)^2}{\rho_3^2}\right] \tag{3.4.4.16}$$

Now we shall solve the new version of the first system where we will try to form an equation for ρ_3 only. Due to the existence of the first integral γ, this will be second order equation. To recognize it better, we introduce a new independent variable z, given as

$$x = \frac{4\sqrt{z}}{\left(1+\sqrt{z}\right)^2} \tag{3.4.4.17}$$

a new dependent variable V given by

$$\rho_3 = \frac{z}{V}\frac{dV}{dz} - \frac{V}{2(z-1)} - \frac{1}{2} + \frac{z}{2V(z-1)} \tag{3.4.4.18}$$

It is not very tedious to show that V satisfies Painlevé-VI equations with parameters $(\alpha, \beta, \gamma, \delta)$ in the notation of [141] or $(\frac{1}{8}, -\frac{1}{8}, \gamma, \frac{1}{2}(1-2\gamma))$ in the notation of [108] . Thus, we see the equation for the conformal factor

$$\Theta = \frac{\rho_1\rho_2\rho_3}{x(1-x)}\frac{dx}{dt} \tag{3.4.4.19}$$

has the Painlevé property, but it also contains the function $x(\tau)$ which has a natural boundary. The choice of conformal factor is equivalent to a gauge choice to make the Ricci scalar vanish.

Now we have found the general solution of the metric (3.4.4.5) inside the top circle of the figure (the shaded region of A). Also note that ASD Bianchi-IX metrics are not always diagonal in the chosen invariant basis of 1-forms. We can always adjust the conformal factor Θ in order to make this ASD Bianchi type metric to become Einstein. This would constitute a metric for the region $A \cap B$, which are quaternionic Kähler type of metrics and are diagonal in the basis.

The solution for the conformal factor was found in [142] with $\gamma = \dfrac{1}{8}$ and writing down the desired condition as a set of equations on Θ and finally solve it. After a little algebra and once all the dusts get settled, we get $\Theta = \dfrac{N}{D^2}$, with

$$\begin{aligned} N &= 2\rho_1\rho_2\rho_3(4x\rho_1\rho_2\rho_3 + P) \\ P &= x\left(\rho_1^2 + \rho_2^2\right) - (1-4\rho_3^2)\left(\rho_2^2 - (1-x)\rho_1^2\right) \\ D &= x\rho_1\rho_2 + 2\rho_3\left(\rho_2^2 - (1-x)\rho_1^2\right) \end{aligned} \tag{3.4.4.20}$$

Since the equation for ρ_3 is a 2nd order differential equation, the metric depends on two arbitrary constants. Particularly it is worth mentioning that there exists ASD Einstein metrics on S^4, which with appropriate choices of field equations fill in the general left-invariant metric on S^3 similar to the case of a 4-dim hyperbolic metric that fills the round metric on S^3.

3.5 Taub-NUT as a Bertrand space-time with Magnetic Fields

The Taub-NUT [35] is an exact solution of Einstein's equations, found by Abraham Huskel Taub (1951), and extended to a larger manifold by E. Newman, T. Unti and L. Tamburino (1963). It is a gravitational anti-instanton with corresponding $SU(2)$ gauge fields, frequently studied for its geodesics which approximately describe the motion of well separated monopole-monopole interactions. As a dynamical system it exhibits spherically symmetry, with geodesics admitting Kepler-type symmetry, implying first-integrals such as the angular momentum and Runge-Lenz vectors respectively. Witten's prescription [143] realized Taub-NUT space as a hyper-Kahler quotient using T-duality. This construction has a natural interpretation in terms of D-branes [144], serving as an important example in string theory.

The Bertrand space-time metric, formulated by V. Perlick [38] is also spherically symmetric

$$ds^2 = h(\rho)^2 d\rho^2 + \rho^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) - \frac{dt^2}{\Gamma(\rho)}, \tag{3.5.1}$$

derived from Bertrand's Theorem, describing stable and closed geodesics with periodic orbits. Upon comparison, Euclidean Bertrand spaces and Taub-NUT spaces, appear quite similar apart from magnetic monopole and dipole interaction of the Taub-NUT. This implies dynamical similarities, manifested through similar first-integrals characterizing their motion. It also implies that Taub-NUT possibly exhibits Kepler-Hooke configuration duality.

Consequently, we try to find first-integrals similar to those associated with central-force motion under potentials involved in Bertrand's Theorem: the angular momentum and Laplace-Runge-Lenz vector. Since we are interested in the dynamical aspects of Taub-NUT spaces, our attention is directed toward geodesics and Killing tensors. Naturally, we will be looking at Killing tensors affiliated with Runge-Lenz-like vector. They obey the equation:

$$\nabla_{(a} K_{b_1) b_2 \ldots b_n} = 0. \tag{3.5.2}$$

Such tensors are the Killing-Stäckel tensors which are symmetric under index permutation. The Killing-Yano tensors are antisymmetric under index permutation, and their square gives the Stäckel tensor, like the antisymmetric tensor whose square gives the Runge-Lenz-like quantity as we shall see. Such Killing tensors exhibit quaternionic algebra, implying a connection to Hyperkähler structures associated with the metric.

3.5.1 Conserved Quantities

In classical mechanics it is important to identify constants of motion called conserved quantities or first-integrals of the system. In the theory of integrable systems, all first-integrals are in involution or commute with each other within the Poisson Brackets, with at least one integral definitely being available. Given a $n+1$-space-time metric with t as cyclic variable:

$$ds^2 = g_{ij}(\boldsymbol{x}) dx^i dx^j + g_{tt}(\boldsymbol{x}) dt^2 + 2 g_{it}(\boldsymbol{x}) dx^i dt,$$

parameterized as $t = \tau$, we will have the Lagrangian and a conserved quantity q:

$$L_{\dot{t}=1} = \frac{1}{2}\left(g_{ij}(\boldsymbol{x}) \dot{x}^i \dot{x}^j\right) + \frac{1}{2} g_{tt}(\boldsymbol{x}) + g_{it}(\boldsymbol{x}) \dot{x}^i, \qquad q = \left(\frac{\partial L}{\partial \dot{t}}\right)_{\dot{t}=1} = g_{tt} + g_{it}\dot{x}^i. \tag{3.5.1.1}$$

The Hamiltonian is given by the Legendre transform $H = \sum_{k \neq t} \frac{\partial L}{\partial \dot{x}^k} \dot{x}^k - L$, so that:

$$H = \frac{1}{2} g^{ij}(\boldsymbol{x}) \left(p_i - g_{it}(\boldsymbol{x})\right)\left(p_j - g_{jt}(\boldsymbol{x})\right) - \frac{1}{2} g_{tt}(\boldsymbol{x}), \qquad p_i = \frac{\partial L}{\partial \dot{x}^i}. \tag{3.5.1.2}$$

In Hamiltonian dynamics, a conserved quantity Q commutes with the Hamiltonian H, a first integral resulting from time translation invariance, within the Poisson Brackets:

$$\{Q, H\} = 0. \tag{3.5.1.3}$$

However, this prescription is not gauge covariant for systems with gauge interactions. To better understand why, consider the following metric with scalar potential $U(\boldsymbol{x})$:

$$ds^2 = \delta_{ij} dx^i dx^j - \frac{1 + 2U(\boldsymbol{x})}{m} dt^2,$$

where t is cyclical. Under the parametrization $t = \tau$, the Lagrangian, Hamiltonian and Hamilton's dynamical equations for particles in presence of scalar potentials is given by:

$$L_{\dot{t}=1} = \frac{m}{2}\dot{\boldsymbol{x}}^2 - U(\boldsymbol{x}),$$

$$H = \frac{1}{2m}\boldsymbol{p}^2 + U(\boldsymbol{x}) \quad \Rightarrow \quad \begin{cases} \dot{\boldsymbol{x}} = & \dfrac{\partial H}{\partial \boldsymbol{p}} = \dfrac{\boldsymbol{p}}{m} \\ \dot{\boldsymbol{p}} = - & \dfrac{\partial H}{\partial \boldsymbol{x}} = -\boldsymbol{\nabla} U(\boldsymbol{x}) \end{cases} .$$

For this system without magnetic fields, the fundamental brackets are:

$$\left\{x^i, p_j\right\} = \delta^{ij}, \qquad \left\{x^i, x^j\right\} = \{p_i, p_j\} = 0.$$

Now, for charged particles in U(1) gauge fields from magnetic dipoles alone, without scalar potential, the metric is:

$$ds^2 = \delta_{ij} dx^i dx^j - \frac{1}{m}\left(dt^2 - 2A_k(\boldsymbol{x}) dx^k dt\right),$$

so the corresponding Lagrangian and Hamiltonian for $\dot{t} = 1$ are given by:

$$L_{\dot{t}=1} = \frac{1}{2}\left(m\dot{\boldsymbol{x}}^2 - \dot{t}^2 + 2\boldsymbol{A}(\boldsymbol{x}).\dot{\boldsymbol{x}}\dot{t}\right), \qquad q = \left(\frac{\partial L}{\partial \dot{t}}\right)_{\dot{t}=1} = 1 - \boldsymbol{A}(\boldsymbol{x}).\dot{\boldsymbol{x}},$$

$$\therefore \qquad H \approx \frac{1}{2m}\left(\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})\right)^2.$$

For charged particles in the presence of magnetic monopole and dipole U(1) gauge fields without scalar potential, the metric is:

$$ds^2 = \delta_{ij} dx^i dx^j - \frac{1}{m}\left(dt - A_k(\boldsymbol{x}) dx^k\right)^2, \tag{3.5.1.4}$$

so the corresponding Lagrangian and Hamiltonian for $\dot{t} = 1$ are given by:

$$L_{\dot{t}=1} = \frac{1}{2}\left[m\dot{\boldsymbol{x}}^2 - (1 - \boldsymbol{A}(\boldsymbol{x}).\dot{\boldsymbol{x}})^2\right], \qquad q = \left(\frac{\partial L}{\partial \dot{t}}\right)_{\dot{t}=1} = 1 - \boldsymbol{A}(\boldsymbol{x}).\dot{\boldsymbol{x}},$$

$$\therefore \qquad H = \frac{1}{2m}\left(\mathbf{p} - q\boldsymbol{A}(\boldsymbol{x})\right)^2.$$

Now let us consider a Kaluza-Klein modification of this space-time, such that we include another cyclical co-ordinate ψ that is periodic along with magnetic field components coupled with it. This would result in a $4+1$ space-time from a $3+1$ one given by:

$$ds^2 = \delta_{ij} dx^i dx^j + \frac{1}{m}\left(d\psi + A_k(\boldsymbol{x}) dx^k\right)^2 - (1 + 2V(\boldsymbol{x}))\, dt^2, \tag{3.5.1.5}$$

so the Lagrangian and Hamiltonian for $\dot{t} = 1$, ignoring constant additive terms are:

$$\begin{aligned} L = \frac{1}{2}\left[m\dot{\boldsymbol{x}}^2 + \left(\dot{\psi} + \boldsymbol{A}(\boldsymbol{x}).\dot{\boldsymbol{x}}\right)^2\right] - V(r), \qquad q = \frac{\partial L}{\partial \dot{\psi}} = \dot{\psi} - \boldsymbol{A}(\boldsymbol{x}), \\ \therefore \qquad H = \frac{1}{2m}\left(\mathbf{p} - q\boldsymbol{A}(\boldsymbol{x})\right)^2 + V(r). \end{aligned} \tag{3.5.1.6}$$

where q is a conserved charge. The corresponding Hamilton's equations are:

$$\dot{\boldsymbol{x}} = \frac{\partial H}{\partial \boldsymbol{p}} = \frac{\boldsymbol{p} - q\boldsymbol{A}}{m}, \qquad \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{x}} = \frac{q}{m}\left(\boldsymbol{\nabla A}\right).\left(\boldsymbol{p} - q\boldsymbol{A}\right) - \boldsymbol{\nabla} V. \tag{3.5.1.7}$$

Since the potentials are gauge dependent ($\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\Lambda$), the momenta therefore must be so as well ($\boldsymbol{p} \to \boldsymbol{p} + q\boldsymbol{\nabla}\Lambda$). Then, we must write gauge invariant momenta and express the Hamiltonian in its gauge invariant form.

$$H = \frac{\boldsymbol{\Pi}^2}{2} + V(r), \qquad \boldsymbol{\Pi} = \boldsymbol{p} - q\boldsymbol{A}. \tag{3.5.1.8}$$

Any function and partial derivative operators in gauge invariant forms can be written as:

$$\begin{aligned}
&f(\boldsymbol{x},\boldsymbol{p}) \longrightarrow f(\boldsymbol{x},\boldsymbol{\Pi}) \\
&\frac{\partial}{\partial x^i} \quad \longrightarrow \quad \frac{\partial \Pi^j}{\partial x^i}\frac{\partial}{\partial \Pi^j} + \frac{\partial}{\partial x^i} = -q\partial_i A_j \frac{\partial}{\partial \Pi^j} + \frac{\partial}{\partial x^i} \\
&\frac{\partial}{\partial p^i} \quad \longrightarrow \quad \frac{\partial \Pi^j}{\partial p^i}\frac{\partial}{\partial \Pi^j} + \frac{\partial}{\partial p^i} = \frac{\partial}{\partial \Pi^i} \qquad (\text{ No explicit dependence on } \boldsymbol{p}\ )
\end{aligned}$$

with which the fundamental brackets become:

$$\left\{x^i, \Pi_j\right\} = \delta^{ij}, \qquad \left\{x^i, x^j\right\} = 0, \qquad \left\{\Pi_i, \Pi_j\right\} = -qF_{ij}, \tag{3.5.1.9}$$

where it is interesting to note that the new Poisson Brackets between the gauge covariant momenta are non-zero, as opposed to the usual case. This is a classical analogue of Ricci-identity (in the absence of torsion). We can furthermore redefine the Poisson Brackets as:

$$\{f, g\} = \frac{\partial f}{\partial \boldsymbol{x}} \cdot \frac{\partial g}{\partial \boldsymbol{\Pi}} - \frac{\partial f}{\partial \boldsymbol{\Pi}} \cdot \frac{\partial g}{\partial \boldsymbol{x}} - qF_{ij}\frac{\partial f}{\partial \boldsymbol{\Pi}} \cdot \frac{\partial g}{\partial \boldsymbol{\Pi}}.$$

Now that we have redefined the Poisson Brackets to make Hamiltonian dynamics manifestly gauge invariant in the modified bracket, we can proceed to analyze the conserved quantities in a general gauge invariant form. This is done by the Holten Algorithm as shown in [145] and [146] discussed later as we shall see.

A dynamical-systems description of Taub-NUT

The Euclidean Taub-NUT metric as shown in [35] is given by:

$$\begin{aligned}
&ds^2 = f(r)\left\{dr^2 + r^2\left(d\theta^2 + \sin^2\theta\ d\phi^2\right)\right\} + g(r)\left(d\psi + \cos\theta d\phi\right)^2, \\
&\quad \text{where} \qquad f(r) = 1 + \frac{4M}{r}, \qquad g(r) = \frac{(4M)^2}{1 + \frac{4M}{r}}.
\end{aligned} \tag{3.5.1.10}$$

For later reference, taking $d\tilde{s}^2 = \dfrac{ds^2}{4M}$ we shall re-write the above metric into this form :

$$\begin{aligned}
&d\tilde{s}^2 = V(r)\ \delta_{ij}\ dx^i dx^j + V^{-1}(r)\left(d\psi + \boldsymbol{A}.d\boldsymbol{x}\right)^2, \\
&\quad \text{where} \qquad V(r) = \frac{1}{4M} + \frac{1}{r}, \qquad \boldsymbol{A}.d\boldsymbol{x} = \cos\theta d\phi.
\end{aligned} \tag{3.5.1.11}$$

We now consider the geodesic flows of the generalized Taub-NUT metric given by (3.5.1.10), for which we can compose the Lagrangian:

$$\mathcal{L} = \frac{1}{2}f(r)\left\{\dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)\right\} + \frac{1}{2}g(r)\left(\dot{\psi} + \cos\theta\dot{\phi}\right)^2. \tag{3.5.1.12}$$

We can further re-write the Lagrangian (3.5.1.12) into 3-dimensional form with a potential, as in (3.5.1.6), independent of the ψ as:

$$\mathcal{L} = \frac{1}{2}f(r)|\dot{\boldsymbol{x}}|^2 + \frac{1}{2}g(r)\left(\dot{\psi} + \boldsymbol{A}.\dot{\boldsymbol{x}}\right)^2 - U(r),$$

where the momentum can be written as:

$$\boldsymbol{p} = \frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{x}}} = f(r)\dot{\boldsymbol{x}} + q\boldsymbol{A}, \qquad\qquad \boldsymbol{\Pi} = f(r)\dot{\boldsymbol{x}} = \boldsymbol{p} - q\boldsymbol{A}. \tag{3.5.1.13}$$

Spaces with the metric (3.5.1.10) exhibit $SU(2)\times U(1)$ isometry group. Given that we have at least 2 cyclical variables ψ and ϕ, we will have the following 4 Killing vectors given by:

$$\begin{aligned}
D_0 &= \ \partial_\psi,\\
D_1 &= -\sin\phi\ \partial_\theta - \cos\phi\cot\theta\ \partial_\phi + \frac{\cos\phi}{\sin\theta}\ \partial_\psi,\\
D_2 &= \quad\ \cos\phi\ \partial_\theta - \sin\phi\cot\theta\ \partial_\phi + \frac{\sin\phi}{\sin\theta}\ \partial_\psi,\\
D_3 &= \ \partial_\phi.
\end{aligned}$$

where D_0 commutes with all other killing vectors, while D_1, D_2, D_3 exhibit the $SU(2)$ Lie algebra given by $[D_i, D_j] = -\varepsilon_{ij}{}^k D_k$. Since ψ is cyclic, we have a conserved quantity:

$$q = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = g(r)\left(\dot{\psi} + \cos\theta\ \dot{\phi}\right) = g(r)\left(\dot{\psi} + \boldsymbol{A}.\dot{\boldsymbol{x}}\right) = const, \tag{3.5.1.14}$$

known as the relative electric charge. Using (3.5.1.13), the symplectic 2-form ω and energy $\mathcal{E}$ for the Taub- NUT system in Cartesian co-ordinates are:

$$\omega = \sum_{i=1}^{3} d\Pi_i\wedge dx^i = \sum_{i=1}^{3} d\left(p_i - qA_i(\boldsymbol{x})\right)\wedge dx^i = \sum_{i=1}^{3} dp_i\wedge dx^i + q\sum_{i,j} F_{ij}dx^i\wedge dx^j,$$

$$\therefore \qquad \omega = \frac{1}{2}\left(\omega_0 + qF(\boldsymbol{x})\right)_{jk} dx^j\wedge dx^k = \sum_{i=1}^{3} dp_i\wedge dx^i - \frac{q}{2r^3}\sum_{i,j,k}\varepsilon_{ijk}x^k\ dx^i\wedge dx^j, \tag{3.5.1.15}$$

$$\mathcal{H} = \frac{|\boldsymbol{\Pi}|^2}{2f(r)} + \frac{q^2}{2g(r)} = \mathcal{E}, \qquad\qquad F_{ij}(\boldsymbol{x}) = -\sum_k \varepsilon_{ijk}\frac{x^k}{r^3}. \tag{3.5.1.16}$$

Consequently, the Hamilton's equations are given by:

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \{\boldsymbol{x}, \mathcal{H}\}_\theta = \frac{\boldsymbol{\Pi}}{f(r)},\\
\dot{\boldsymbol{\Pi}} &= \{\boldsymbol{\Pi}, \mathcal{H}\}_\theta = \alpha(r)\frac{\boldsymbol{x}}{r} + \frac{q}{r^3 f(r)}\boldsymbol{x}\times\boldsymbol{\Pi} - \boldsymbol{\nabla}U(r),
\end{aligned} \qquad \text{where} \quad \alpha(r) = \frac{f'(r)}{2\left(f(r)\right)^2}|\boldsymbol{\Pi}|^2 + \frac{g'(r)}{2\left(g(r)\right)^2}. \tag{3.5.1.17}$$

Using these equations, we find angular momentum in presence of magnetic fields to be:

$$\left(\frac{d\boldsymbol{x}}{dt}\times\boldsymbol{\Pi}+\boldsymbol{x}\times\frac{d\boldsymbol{\Pi}}{dt}\right)=\frac{q}{r^3 f(r)}\left[\boldsymbol{x}\times(\boldsymbol{x}\times\boldsymbol{\Pi})\right]=q\left[\frac{(\boldsymbol{x}.\dot{\boldsymbol{x}})\,\boldsymbol{x}}{r^3}-\frac{\dot{\boldsymbol{x}}}{r}\right]=-q\frac{d}{dt}\left(\frac{\boldsymbol{x}}{r}\right),$$

$$\therefore\qquad \frac{d}{dt}\left(\boldsymbol{x}\times\boldsymbol{\Pi}+q\frac{\boldsymbol{x}}{r}\right)=0\qquad\Rightarrow\qquad \boldsymbol{J}=\boldsymbol{x}\times\boldsymbol{\Pi}+q\frac{\boldsymbol{x}}{r}. \tag{3.5.1.18}$$

The cyclic variable allows reduction of the geodesic flow on $T(\mathbb{R}^4-\{0\})$ to a system on $T(\mathbb{R}^3-\{0\})$. The reduced system's rotational invariance implies it must have a conserved energy, angular momentum and vector $\boldsymbol{K}$ analogous to the Laplace-Runge-Lenz vector:

$$\mathcal{H}=\frac{1}{2}\frac{\boldsymbol{\Pi}^2}{f(r)}+\left(\frac{1}{2}\frac{q^2}{g(r)}+U(r)\right)=\frac{1}{2}\frac{\boldsymbol{\Pi}^2}{f(r)}+W(r), \tag{3.5.1.19}$$

$$\boldsymbol{J}=\boldsymbol{x}\times\boldsymbol{\Pi}+q\frac{\boldsymbol{x}}{r}, \tag{3.5.1.20}$$

$$\boldsymbol{K}=\frac{1}{2}K_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=\boldsymbol{\Pi}\times\boldsymbol{J}+\left(\frac{q^2}{4m}-4mE\right)\frac{\boldsymbol{x}}{r}. \tag{3.5.1.21}$$

This concludes the detailing of conserved quantities of the Taub-NUT from a dynamical systems perspective. Now we shall proceed to consider a systematic analytic process that describes conserved quantities as power series expansions of momenta.

Holten Algorithm description

One way of analytically obtaining conserved quantities that are polynomials in momenta is by writing them in a power series expansion involving the gauge invariant momenta:

$$Q=C^{(0)}(\mathbf{r})+C^{(1)}_i(\mathbf{r})\Pi^i+\frac{1}{2!}C^{(2)}_{ij}(\mathbf{r})\Pi^i\Pi^j+\frac{1}{3!}C^{(3)}_{ijk}(\mathbf{r})\Pi^i\Pi^j\Pi^k+.... \tag{3.5.1.22}$$

where all the coefficients of momenta power series are symmetric under index permutation. Applying this to eq (3.5.1.3), we can obtain the relations for each coefficient by matching the appropriate product series of momenta for both the terms.

$$\{Q,H\}=\sum_n\left[\left\{C^{(n)}_{\{i\}}\prod_{\{i\}}\Pi^k,\Pi^j\right\}\Pi_j+\left\{C^{(n)}_{\{i\}}\prod_{\{i\}}\Pi^k,V(r)\right\}\right]=0,$$

$$\therefore\qquad \nabla_j C^{(n)}_{\{m\}}\prod_{\{m\}}\Pi^k=qC^{(n+1)}_{\{m\}i}\Big(F^{ij}+\partial_j V(r)\Big)\prod_{(\{m\},k\neq i)}\Pi^k \tag{3.5.1.23}$$

The equations we will get up to the 3rd order setting $C^{(i)}_{\{m\}}=0\quad\forall\quad i\geq 3$ are:

$$\begin{aligned}
&\text{order 0:} && 0=C^{(1)}_m\partial_m\left(V(r)\right)\\
&\text{order 1:} && \nabla_i C^{(0)}=qF_{ij}C^{(1)}_j+C^{(2)}_{ij}\partial_j\left(V(r)\right)\\
&\text{order 2:} && \nabla_i C^{(1)}_j+\nabla_j C^{(1)}_i=q\left(F_{im}C^{(2)m}{}_j+F_{jm}C^{(2)m}{}_i\right)\\
&\text{order 3:} && \nabla_i C^{(2)}_{jk}+\nabla_k C^{(2)}_{ij}+\nabla_j C^{(2)}_{ki}=0
\end{aligned} \tag{3.5.1.24}$$

Now we will turn our attention to some familiar conserved quantities.

Some basic Killing Tensors

Using the above relations for the various terms, we will now look at some familiar Killing tensors that we have already studied in classical mechanics. (for details, see Appendix 6.2)

Angular Momentum

The conserved quantity that results from the 1st order term of the Holten series alone is:

$$
\begin{aligned}
Q^{(1)} &= C_i^{(1)}\Pi^i = -g_{im}(\vec{x})\varepsilon^m{}_{jk}\theta^k x^j \Pi^i \\
\Rightarrow \qquad \boldsymbol{L}.\boldsymbol{\theta} &= -\left(\varepsilon_{ijk}\Pi^i x^j\right)\theta^k = (\boldsymbol{x}\times\boldsymbol{\Pi})\,.\boldsymbol{\theta} \\
\therefore \qquad \boldsymbol{L} &= \boldsymbol{x}\times\boldsymbol{\Pi}
\end{aligned}
\tag{3.5.1.25}
$$

This eventually becomes the conserved quantity known as the angular momentum.

Laplace-Runge-Lenz vector

On the other hand, the conserved quantity from the 2nd order term of the series alone is:

$$
Q^{(2)} = \frac{1}{2}C_{ij}^{(2)}\Pi^i\Pi^j = \left\{\left|\boldsymbol{\Pi}\right|^2(\boldsymbol{n}.\boldsymbol{x}) - (\boldsymbol{\Pi}.\boldsymbol{x})(\boldsymbol{\Pi}.\boldsymbol{n})\right\} \;=\; \left\{\left|\boldsymbol{\Pi}\right|^2\boldsymbol{x} - (\boldsymbol{\Pi}.\boldsymbol{x})\,\boldsymbol{\Pi}\right\}.\boldsymbol{n} = \boldsymbol{N}.\boldsymbol{n}.
$$

$$
\therefore \qquad \boldsymbol{N} = \left\{\left|\boldsymbol{\Pi}\right|^2\boldsymbol{x} - (\boldsymbol{\Pi}.\boldsymbol{x})\,\boldsymbol{\Pi}\right\} = \boldsymbol{\Pi}\times(\boldsymbol{x}\times\boldsymbol{\Pi}) \tag{3.5.1.26}
$$

This quantity is a term contained in another conserved quantity known as the Laplace-Runge-Lenz vector. Having found the two familiar types of conserved quantities, we can now proceed to see what it looks like for the Taub-NUT metric.

Holten algorithm for Taub-NUT

Now, for the Taub-NUT metric, we have (3.5.1.19) giving the Hamiltonian. This can be written in dimensionally reduced form as:

$$
\mathcal{H} = \frac{1}{2}|\boldsymbol{\Pi}|^2 + f(r)W(r), \qquad\qquad W(r) = U(r) + \frac{q^2}{2h(r)} + \frac{\mathcal{E}}{f(r)} - \mathcal{E}.
$$

From this Hamiltonian, after setting all higher orders $C_{ij}^{(2)} = C_{ijk}^{(3)} = 0$, we get the modified 1st and 2nd order equations to be the following:

$$
\begin{aligned}
&\text{order 1:} \qquad \partial_i C^{(0)} = qF_{ij}C^{(1)j}, \\
&\text{order 2:} \qquad \nabla_i C_j^{(1)} + \nabla_j C_i^{(1)} = 0.
\end{aligned}
\tag{3.5.1.27}
$$

The constraint equation of the 2nd order of (3.5.1.27) gives us:

$$
C_i^{(1)} = g_{im}(\boldsymbol{x})\varepsilon^m{}_{jk}\theta^j x^k,
$$

$$
\partial_i C^{(0)} = \frac{q}{r^3}\varepsilon_{ijk}\,\varepsilon^j{}_{nm}x^k\theta^m x^n \quad\equiv\quad \frac{q}{r^3}\left[\boldsymbol{x}\times(\boldsymbol{\theta}\times\boldsymbol{x})\right]_i = \frac{q}{r^3}\left[r^2\boldsymbol{\theta} - (\boldsymbol{x}.\boldsymbol{\theta})\,\boldsymbol{x}\right]_i,
$$

$$
\therefore \qquad \nabla_i C^{(0)} = q\left(\frac{\theta_i}{r} - \frac{(\boldsymbol{x}.\boldsymbol{\theta})\,x_i}{r^3}\right) \qquad \Rightarrow \qquad C^{(0)} = q\theta_i\frac{x^i}{r}. \tag{3.5.1.28}
$$

Thus, we have the overall solution, and the corresponding conserved quantity:

$$Q \equiv J_k\theta^k = C^{(0)} + C_i^{(1)}\Pi^i = \left(-g_{im}(\vec{x})\varepsilon^m{}_{jk}x^j\Pi^i + q\frac{x_k}{r}\right)\theta^k,$$

$$\therefore \qquad \boldsymbol{J}.\boldsymbol{\theta} = \left(\boldsymbol{x}\times\boldsymbol{\Pi} + q\frac{\boldsymbol{x}}{r}\right).\boldsymbol{\theta} \qquad \Rightarrow \qquad \boldsymbol{J} = \boldsymbol{x}\times\boldsymbol{\Pi} + q\frac{\boldsymbol{x}}{r}. \tag{3.5.1.29}$$

However, if we explore upto the 2nd order, setting $C_{ij}^{(2)} \neq 0$, we will return to the equations in (3.5.1.24). For the 3rd order equation, the solution for $C_{ij}^{(2)}$ is given by (6.2.2.3), so that:

$$C_{ij}^{(2)} = \left(2g_{ij}(\boldsymbol{x})n_k - g_{ik}(\boldsymbol{x})n_j - g_{kj}(\boldsymbol{x})n_i\right)x^k. \tag{3.5.1.30}$$

Eventually the other co-efficients are given by:

$$\nabla_{(i}C_{j)}^{(1)} = q\left(F_{ik}C_{kj}^{(2)} + F_{jk}C_{ki}^{(2)}\right),$$

$$F_{ik}C_{kj}^{(2)} = -2\varepsilon_{ijn}\frac{x^n}{r^3}\underbrace{(n_m x^m)}_{\boldsymbol{n}.\boldsymbol{x}} + \underbrace{\varepsilon_{ikn}\frac{x^k x^n}{r^3}n_j}_{0} + \underbrace{\varepsilon_{ikn}n^k x^n}_{(\boldsymbol{n}\times\boldsymbol{x})_i}\frac{x_j}{r^3}$$

$$\therefore \qquad \nabla_i C_j^{(1)} + \nabla_j C_i^{(1)} = q\left\{\frac{x_j}{r^3}(\boldsymbol{n}\times\boldsymbol{x})_i + \frac{x_i}{r^3}(\boldsymbol{n}\times\boldsymbol{x})_j\right\}$$

Here, one can choose to insert extra terms:

$$\nabla_i C_j^{(1)} + \nabla_j C_i^{(1)} = -q\left\{\nabla_j\left(\frac{\varepsilon_{ikm}n^k x^m}{r}\right) + \nabla_i\left(\frac{\varepsilon_{jkm}n^k x^m}{r}\right)\right\}.$$

Thus, we can easily see which term on the RHS corresponds to what on the LHS, allowing us to solve for the 1st order and zeroth order coefficients from (3.5.1.24) :

$$C_i^{(1)} = -\frac{q}{r}g_{im}(\boldsymbol{x})\varepsilon^m{}_{jk}n^k x^j \tag{3.5.1.31}$$

$$\nabla_i C^{(0)} = -q^2\left(\frac{n_i}{r^2} - \frac{(\boldsymbol{x}.\boldsymbol{n})\,x_i}{r^4}\right) + \left\{2\,(\boldsymbol{n}.\boldsymbol{x})\,\delta_{ij} - n_i x_j - x_i n_j\right\}\partial_j\left(f(r)U(r) + q^2\frac{f(r)}{2g(r)} + \mathcal{E} - \mathcal{E}f(r)\right).$$

In the case of the generalized Taub-NUT metric, the most general potentials admitting a Runge-Lenz vector are of the form:

$$U(r) = \frac{1}{f(r)}\left(\frac{q^2}{2r^2} + \frac{\beta}{r} + \gamma\right) - \frac{q^2}{2g(r)} + \mathcal{E} \tag{3.5.1.32}$$

$$\nabla_i C^{(0)} = \beta\left(\frac{n_i}{r} - \frac{(\boldsymbol{n}.\boldsymbol{x})\,x_i}{r^3}\right) \qquad\qquad C^{(0)} = \beta n_i\frac{x^i}{r}. \tag{3.5.1.33}$$

For integrability, we require the commutation relation:

$$[\partial_i, \partial_j]\,C^{(0)} = 0 \quad \Rightarrow \quad \Delta\left(f(r)W(r) - \frac{q^2g^2}{2r^2}\right) = 0 \quad \Rightarrow \quad f(r)W(r) - \frac{q^2g^2}{2r^2} = \frac{\beta}{r} + \gamma \qquad \beta, \gamma \in \mathbb{R}.$$

Thus, this overall conserved quantity is given as:

$$Q \equiv R_k\theta^k = C^{(0)} + C_i^{(1)}\Pi^i + C_{ij}^{(2)}\Pi^i\Pi^j,$$

$$\boldsymbol{R}.\boldsymbol{n} = \left(\boldsymbol{\Pi}\times(\boldsymbol{x}\times\boldsymbol{\Pi}) - \frac{q}{r}\boldsymbol{x}\times\boldsymbol{\Pi} + \beta\frac{\boldsymbol{x}}{r}\right).\boldsymbol{n} \qquad \Rightarrow \qquad \boldsymbol{R} = \boldsymbol{\Pi}\times\boldsymbol{J} + \beta\frac{\boldsymbol{x}}{r} \tag{3.5.1.34}$$

Now we will take a detour to look at some details regarding the Runge-Lenz vector.

3.5.2 Bertrand space-time dualities

In Newtonian mechanics, there are only two potentials allowing stable, closed and periodic orbits: Hooke's Oscillator ($V(r) = ar^2 + b$), and Kepler's Orbital Motion ($\Gamma(r) = \frac{a}{r} + b$) potentials. There is a relativistic analogue, given by the corresponding metrics in [38], describing spherically symmetric and static space-time, with bounded and periodic trajectories. The Taub-NUT is one example of a spherically symmetric space-time. Naturally, one would compare it with the Euclidean Bertrand space-time (BST) metric with magnetic fields.

Bertrand space-times with magnetic fields

The Bertrand space-time metric is given by (3.5.1). If we take the Euclidean version and include magnetic monopole and dipole interaction terms, then the metric becomes like (3.5.1.4) as:

$$ds^2 = h(\rho)^2 d\rho^2 + \rho^2 \left(d\theta^2 + \sin^2\theta\ d\phi^2\right) + \frac{1}{\Gamma(\rho)}\left(dt + A_i dx^i\right)^2 \tag{3.5.2.1}$$

If we recall, the Taub-NUT metric was given by (3.5.1.10). To see how they are comparable, we shall attempt a co-ordinate map.

$$f(r)\ dr^2 = h(\rho)^2 d\rho^2 \qquad f(r)r^2 = \rho^2 \qquad g(r) = \frac{1}{\Gamma(\rho)} \qquad t = \psi + k$$

$$\Rightarrow \qquad \frac{dr}{r} = \frac{h(\rho)\ d\rho}{\rho} \qquad \Rightarrow \qquad r = r_0\ e^{\int d\rho\ \frac{h(\rho)}{\rho}}$$

Thus, we can suppose that Taub-NUT metric resembles Bertrand space-time with magnetic fields. We can also proceed the other way around, starting with the generalized Taub-NUT metric and proceeding toward Bertrand space-times by applying appropriate potential power laws shown in [147]. Thus, like the BSTs, there are two Taub-NUT configurations:

1. Hooke's Oscillator configuration:

$$f_{\mathcal{O}}(r) = ar^2 + b \qquad g_{\mathcal{O}}(r) = \frac{r^2\left(ar^2 + b\right)}{cr^4 + dr^2 + 1}.$$

2. Kepler's orbital configuration:

$$f_{\mathcal{K}}(r) = \frac{a + br}{r} \qquad g_{\mathcal{K}}(r) = \frac{r\left(a + br\right)}{cr^2 + dr + 1}.$$

Evidence for the duality between these two configurations of the metric can be clearly demonstrated. To study Taub-NUT space duality, we confine motion to a cone ($\theta = const$). This is permissible because of the conserved angular momentum (3.5.1.18), for which [35, 147, 148]

$$\boldsymbol{J}.\boldsymbol{e^r} = \left|\boldsymbol{J}\right| \cos\theta = const \qquad \Rightarrow \qquad \theta = const \tag{3.5.2.2}$$

This allows us to reduce the problem to 2-dimensions by rendering θ a constant co-ordinate, allowing us to write the metric as:

$$ds^2 = f(r)\left(dr^2 + r^2\alpha^2\ d\phi^2\right) + g(r)\left(d\psi + \beta\ d\phi\right)^2 \qquad \alpha = \sin\theta, \beta = \cos\theta \tag{3.5.2.3}$$

We shall represent the co-ordinates as $Z = x + iy$, $\xi = X + iY$, where $|Z| = r\cos\frac{\theta}{2}$ and perform Bohlin's transformation [149] of the Oscillator metric ($Z \to \xi = Z^2$) (see Appendix

6.3). This requires the complex co-ordinates defined for self-dual Euclidean spaces [118], where ($\theta = const$) (3.5.2.2):

$$Z = x + iy = |Z|\exp\left[\frac{i}{2}(\psi+\phi)\right] \qquad \xi = X + iY = |\xi|\exp\left[\frac{i}{2}(\chi+\Phi)\right] \tag{3.5.2.4}$$

$$Z \to \xi = Z^2 = |Z|^2\exp\left[i(\psi+\phi)\right] \qquad \Rightarrow \qquad \phi \to \Phi = 2\phi,\ \psi \to \chi = 2\psi \tag{3.5.2.5}$$

$$\left(ds^2\right)_{\mathcal{O}} = \left(a|Z|^2+b\right)|dZ|^2 + \frac{|Z|^2(a|Z|^2+b)}{c|Z|^4+d|Z|^2+1}\left(d\psi+\beta\ d\phi\right)^2 \tag{3.5.2.6}$$

$$\begin{gathered}\left(a|Z|^2+b\right)|dZ|^2 + \frac{|Z|^2(a|Z|^2+b)}{c|Z|^4+d|Z|^2+1}\left(d\psi+\beta\ d\phi\right)^2 \\ Z\to\xi=Z^2 \Big\updownarrow \phi\to\Phi=2\phi,\ \psi\to\chi=2\psi \\ \frac{1}{4}\left\{\frac{a|\xi|+b}{|\xi|}|d\xi|^2 + \frac{|\xi|\left(a|\xi|+b\right)}{c|\xi|^2+d|\xi|+1}\left(d\chi+\beta\ d\Phi\right)^2\right\}\end{gathered}$$

Then we can compare with the Kepler system in presence of magnetic fields:

$$\left(ds^2\right)_{\mathcal{K}} = \frac{b|Z|+a}{|Z|}|dZ|^2 + \frac{|Z|\left(b|Z|+a\right)}{c|Z|^2+d|Z|+1}\left(d\chi+\beta\ d\Phi\right)^2 \tag{3.5.2.7}$$

showing that aside from a factor of $\frac{1}{4}$, a variable swap $a \leftrightarrow b$ completes the transformation, and thus, the two configurations of Taub-NUT are also related via Bohlin's transformation like Bertrand space-time. For various settings of the constants, one can get different configurations of space-time, as shall be described in the following table.

Kepler-Oscillator duality

In the study of central force problem, we learn that the Kepler and Oscillator systems are dual to each other according a duality map demonstrated in [30] and [36]. This is summed up in Bertrand's theorem which describes them as the only systems with stable, closed and periodic orbits. Thus, curved Bertrand space-times are classified as Type I and Type II, representing Kepler and Oscillator systems respectively.

If we start with the 2-dimensional simple harmonic oscillator described by $\ddot{x}^i = -\omega^2 x^i$, we are reminded of a conserved tensorial quantity, known as the Fradkin tensor:

$$T^{ij} = p^i p^j + \kappa x^i x^j \qquad i,j = 1,2 \tag{3.5.2.8}$$

Any conserved quantity can be obtained by contracting the Fradkin tensor over its two indices by any chosen structure. ie.

$$Q = M_{ij}T^{ij} \tag{3.5.2.9}$$

This quantity is symmetric under index permutation. Its complex counterpart is given by:

$$T_{z^a z^b} = G^{ij}_{z^a z^b} T_{ij} \qquad z^a = \{z, \bar{z}\} \tag{3.5.2.10}$$

$$G_{zz} = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \qquad G_{\bar{z}\bar{z}} = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \qquad G_{z\bar{z}} = G_{\bar{z}z} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.5.2.11}$$

Table 3.1: Systems for various settings ($\mathcal{K}$ - Kepler, $\mathcal{O}$ - Oscillator)

Type	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$	$\boldsymbol{f(r)}$	$\boldsymbol{g(r)}$	**System Name**
$\mathcal{K}$	0	1	1	-2	1	$\dfrac{r^2}{(1-r)^2}$	MIC-Zwangier
$\mathcal{K}$	0	1	0	$-\dfrac{2k}{q^2}$	1	$\dfrac{r^2}{1-\frac{2k}{q^2}r}$	MIC-Kepler
$\mathcal{O}$	0	1	$\dfrac{k}{q^2}$	0	1	$\dfrac{r^2}{1+\frac{k}{q^2}r^4}$	MIC-Oscillator
$\mathcal{K}$	$4m$	1	0	$\dfrac{1}{4m}$	$\dfrac{4m+r}{r}$	$\dfrac{(4m)^2 r}{4m+r}$	Euclidean Taub-NUT

According to the Arnold-Vasiliev duality [150], a co-ordinate transformation and re-parametrization of the first two complex Fradkin tensors will give us the Laplace-Runge-Lenz vector.

$$\boldsymbol{A} = \boldsymbol{p} \times \boldsymbol{L} + \beta \frac{\boldsymbol{x}}{r} \tag{3.5.2.12}$$

In tensorial form, this is written as follows:

$$A_i = \varepsilon_{ikl}\varepsilon^l{}_{jm} p^k x^j p^m + \frac{\beta}{r}\delta_{ij}x^j = x^j \left\{ (\delta_{ij}\delta_{km} - \delta_{ik}\delta_{jm})\, p^k p^m + \frac{\beta}{r}\delta_{ij} \right\} \tag{3.5.2.13}$$

showing that the 1st term can be expressed in a form quadratic in momenta. Since it is essentially a linear combination of Fradkin tensor components, we would prefer it to be symmetric in the momentum indices like its oscillator counterpart. Thus, we can write

$$A_i = x^j \left\{ \frac{1}{2}\left(2\delta_{ij}\delta_{km} - \delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk}\right) p^k p^m + \frac{\beta}{r}\delta_{ij} \right\} \tag{3.5.2.14}$$

Hence, to describe this conserved quantity of the Kepler system, we need tensors that are:

1. quadratic in momenta
2. symmetric under index permutation
3. conserved along geodesics

Our next step will be to explore such tensors in the next subsection.

3.5.3 A review of geometric properties

Since we are considering the Taub-NUT metric defined on a 4-dimensional Euclidean plane, it is worthwhile to verify if it is an instanton as well. Here, we will analyze its geometrical properties exhaustively, verify if the Taub-NUT is an instanton from the curvature components computed from the metric, and also take a look at its topological properties.

A variable transformation $m = 2M$ and $\rho = r - 2M$ of (3.5.1.10) gives the Taub-NUT as:

$$ds^2 = \frac{r+m}{r-m}dr^2 + 4m^2\frac{r-m}{r+m}\left(d\psi + \cos\theta\ d\phi\right)^2 + \left(r^2 - m^2\right)\left(d\theta^2 + \sin^2\theta\ d\phi^2\right).$$

This can be further recast into the form:

$$ds^2 = \frac{r+m}{r-m}dr^2 + 4m^2\frac{r-m}{r+m}\sigma_1^2 + \left(r^2 - m^2\right)\left(\sigma_2^2 + \sigma_3^2\right), \tag{3.5.3.1}$$

where the variables σ_i are essentially solid angle elements in 4-dimensional Euclidean space obeying the following structure equation:

$$d\sigma^i = -\varepsilon^i{}_{jk}\ \sigma^j \wedge \sigma^k, \qquad\qquad \sigma^i = -\frac{1}{r^2}\eta^i_{\mu\nu}x^\mu dx^\nu. \tag{3.5.3.2}$$

Now that we have identified the vierbeins, we will proceed to implement Cartan's method of computing spin connections and the Riemann curvature components. Embedded within them are the $SU(2)$ gauge fields and their corresponding field strengths as we shall see.

Taub-NUT as a Darboux-Halphen system

The Taub-NUT is a special case of self-dual Bianchi-IX metrics [116], which are characterized by the classical Darboux-Halphen system. The self-dual metric and its characteristic system of equations are given by:

$$d\widetilde{s}^2 = \left(\Omega_1\Omega_2\Omega_3\right)d\widetilde{r}^2 + \frac{\Omega_2\Omega_3}{\Omega_1}\left(\sigma_1\right)^2 + \frac{\Omega_3\Omega_1}{\Omega_2}\left(\sigma_2\right)^2 + \frac{\Omega_1\Omega_2}{\Omega_3}\left(\sigma_3\right)^2, \tag{3.5.3.3}$$

$$\begin{aligned} \Omega_1' &= \Omega_2\Omega_3 - \Omega_1\left(\Omega_2 + \Omega_3\right) \\ \Omega_2' &= \Omega_3\Omega_1 - \Omega_2\left(\Omega_3 + \Omega_1\right) \\ \Omega_3' &= \Omega_1\Omega_2 - \Omega_3\left(\Omega_1 + \Omega_2\right) \end{aligned} \qquad (\ )' = \frac{d}{d\widetilde{r}}(\ ). \tag{3.5.3.4}$$

where Ω_i are parameters defined to re-write the Bianchi-IX metric into the form (3.5.3.3) to write self-dual equations. One particular first integral of this system [151, 152] is:

$$Q = \frac{(\Omega_1)^2}{(\Omega_3 - \Omega_1)(\Omega_1 - \Omega_2)} + \frac{(\Omega_2)^2}{(\Omega_1 - \Omega_2)(\Omega_2 - \Omega_3)} + \frac{(\Omega_3)^2}{(\Omega_2 - \Omega_3)(\Omega_3 - \Omega_1)} \tag{3.5.3.5}$$

In case of the Taub-NUT, we need to set $\Omega_2 = \Omega_3 = \Omega \neq \Omega_1 = \Lambda$. This way, we will get the following metric, system of equations and first integral:

$$d\widetilde{s}^2 = \Omega^2\Lambda\ d\widetilde{r}^2 + \Lambda\left[\left(\sigma_2\right)^2 + \left(\sigma_3\right)^2\right] + \frac{\Omega^2}{\Lambda}\left(\sigma_1\right)^2, \tag{3.5.3.6}$$

$$\frac{d\Lambda}{d\widetilde{r}} = \Omega(\Omega - 2\Lambda) \qquad\qquad \frac{d\Omega}{d\widetilde{r}} = -\Omega^2 \tag{3.5.3.7}$$

$$\begin{aligned}\left[\lim_{\Omega_2\to\Omega_3=\Omega} Q\right]_{\Omega_1=\Lambda} &= -\frac{\Lambda^2}{(\Lambda-\Omega)^2} + \frac{1}{\Lambda-\Omega}\left[\lim_{\Omega_2\to\Omega_3=\Omega}\left(\frac{(\Omega_2)^2}{\Omega_2-\Omega_3} - \frac{(\Omega_3)^2}{\Omega_2-\Omega_3}\right)\right] \\ &= -\frac{\Lambda^2}{(\Lambda-\Omega)^2} + \frac{2\Omega}{\Lambda-\Omega} = -1 - \left(\frac{\Omega}{\Lambda-\Omega}\right)^2\end{aligned} \tag{3.5.3.8}$$

Rescaling the radius and solving (3.5.3.7) with suitable constants of integration gives us:

$$d\widetilde{r} = -\frac{dr}{2m\Omega^2} \qquad \frac{d\Omega}{dr} = \frac{1}{2m} \qquad \frac{d}{dr}\left(\frac{\Lambda}{\Omega^2}\right) = -\frac{1}{\Omega^2}\frac{d\Omega}{dr}$$

$$\Omega = \frac{r-m}{2m} \qquad\qquad \Lambda = \frac{r^2-m^2}{4m^2} \tag{3.5.3.9}$$

and rescaling the metric as $d\widetilde{s} = \dfrac{ds}{2m}$ we get the Taub-NUT (3.5.3.1) and conserved quantity:

$$\begin{aligned} ds^2 &= \frac{r+m}{r-m}dr^2 + 4m^2\frac{r-m}{r+m}(\sigma_1)^2 + (r^2-m^2)\left[(\sigma_2)^2 + (\sigma_3)^2\right], \\ \left[\lim_{\Omega_2\to\Omega_3=\Omega} Q\right]_{\Omega_1=\Lambda} &= -1 - \left(\frac{\Omega}{\Lambda-\Omega}\right)^2 = -\frac{r^2-2mr+5m^2}{(r-m)^2}\end{aligned} \tag{3.5.3.10}$$

This concludes another possible symmetry of the Taub-NUT as a member of Bianchi-IX metrics or solutions to Darboux-Halphen systems.

Curvature and anti-self duality

We can now analyze its geometry of the Euclidean space (3.5.3.1), as done for the Bianchi-IX metric in Subsec 3.4.1. The spin-connection matrix according to (3.4.1.4) can be constructed as shown below:

$$\omega = \begin{pmatrix} 0 & -\frac{2m^2}{(r+m)^2}\sigma^1 & -\left(1-\frac{m}{r+m}\right)\sigma^2 & -\left(1-\frac{m}{r+m}\right)\sigma^3 \\ \frac{2m^2}{(r+m)^2}\sigma^1 & 0 & -\frac{m}{r+m}\sigma^3 & \frac{m}{r+m}\sigma^2 \\ \left(1-\frac{m}{r+m}\right)\sigma^2 & \frac{m}{r+m}\sigma^3 & 0 & -\left(1-\frac{2m^2}{(r+m)^2}\right)\sigma^1 \\ \left(1-\frac{m}{r+m}\right)\sigma^3 & -\frac{m}{r+m}\sigma^2 & \left(1-\frac{2m^2}{(r+m)^2}\right)\sigma^1 & 0 \end{pmatrix} \tag{3.5.3.11}$$

If we view the spin connections as a linear combination of self dual and anti-self dual tensors, then we can accordingly separate the self and anti-self dual components as $\omega_{ij} = \omega_{ij}^{(+)} + \omega_{ij}^{(-)}$. To this end, we can split the spin connection matrix (3.5.3.11) into two separate components: the self dual and the anti-self dual parts

$$\begin{aligned}\omega^{(+)} &= -\frac{1}{2}\left(\sigma^1\eta_1 + \sigma^2\eta_2 + \sigma^3\eta_3\right) = -\frac{1}{2}\sigma^i\eta_i \\ \omega^{(-)} &= \left\{\left(\frac{1}{2} - \frac{2m^2}{(r+m)^2}\right)\sigma^1\bar{\eta}_1 - \left(\frac{1}{2} - \frac{m}{r+m}\right)\left(\sigma^2\bar{\eta}_2 - \sigma^3\bar{\eta}_3\right)\right\}\end{aligned} \tag{3.5.3.12}$$

For reference, we have the t'Hooft symbol matrices $\eta^{(\pm)}$, which exhibit the $su(2)$ Lie algebra:

$$[\eta_i, \eta_j] = -2\varepsilon_{ij}{}^k \eta_k$$

The curvature tensor can be decomposed into self and anti-self dual parts $R_{ij} = R_{ij}^{(+)} + R_{ij}^{(-)}$, where according to Cartan's 2nd equation, $R = d\omega + \omega \wedge \omega$. Thus, we can write the spin connections as as a linear combination of self and anti-self dual t'Hooft symbols giving us the self-dual and anti-self dual spin connections described in (3.5.3.12). Consequently, according to (3.5.3.2), the curvature tensor vanishes for the self-dual part:

$$\therefore \quad R^{(+)} = d\omega^{(+)} + \omega^{(+)} \wedge \omega^{(+)} = -\frac{1}{2}\Big(d\sigma^i + \varepsilon^i{}_{jk}\, \sigma^j \wedge \sigma^k\Big)\eta_i = 0 \tag{3.5.3.13}$$

Only the anti-self dual curvature remains, reflecting the Taub-NUT's anti-self dual nature. We make our job easier by writing the spin connection as $\omega^{(-)} = \omega_1^{(-)} + \omega_2^{(-)}$:

$$\omega^{(-)} = \frac{1}{2}\left(\sigma^1\bar{\eta}_1 - \sigma^2\bar{\eta}_2 + \sigma^3\bar{\eta}_3\right) + \left\{-\frac{2m^2}{(r+m)^2}\sigma^1\bar{\eta}_1 + \frac{m}{r+m}\left(\sigma^2\bar{\eta}_2 - \sigma^3\bar{\eta}_3\right)\right\} \tag{3.5.3.14}$$

where one can verify that $\omega_1^{(-)}$ will follow the same rule as $\omega^{(+)}$ in (3.5.3.13).

This allows us to compute the anti-self-dual curvature is given by

$$\begin{aligned} \therefore \quad R^{(-)} = &\frac{2m}{(r+m)^3}\bar{\eta}_1\left(e^0 \wedge e^1 - e^2 \wedge e^3\right) \\ &+ \frac{m}{(r+m)^3}\left\{-\bar{\eta}_2\left(e^0 \wedge e^2 - e^3 \wedge e^1\right) + \bar{\eta}_3\left(e^0 \wedge e^3 - e^1 \wedge e^2\right)\right\} \end{aligned} \tag{3.5.3.15}$$

where we can see from the signs attached to the dual components that the curvature derived from Taub-NUT metric is clearly anti-self dual, as shown in [153]. This also lets us conclude that it is an instanton. To elaborate further, we can show that only $SU(2)_-$ gauge fields are embedded within the spin-connection components as shown below:

$$\omega_{\mu\nu}^{(\pm)} = \eta_{\mu\nu}^{(\pm)k} A_k^{(\pm)} \qquad \Rightarrow \qquad A^{(\pm)i} = \frac{1}{4}\eta_{\mu\nu}^{(\pm)i}\omega_{\mu\nu} \tag{3.5.3.16}$$

$$\begin{aligned} A^{(+)1} &= -\frac{\sigma^1}{2} & \qquad A^{(-)1} &= \left(1 - \frac{4m^2}{(r+m)^2}\right)\frac{\sigma^1}{2} \\ A^{(+)2} &= -\frac{\sigma^2}{2} & \qquad A^{(-)2} &= -\frac{r-m}{r+m}\frac{\sigma^2}{2} \\ A^{(+)3} &= -\frac{\sigma^3}{2} & \qquad A^{(-)3} &= \frac{r-m}{r+m}\frac{\sigma^3}{2} \end{aligned} \tag{3.5.3.17}$$

while the field strengths are given by:

$$R_{\mu\nu}^{(-)} = \eta_{\mu\nu}^{(-)k} F_k^{(-)} \qquad \Rightarrow \qquad F^{(\pm)i} = \frac{1}{4}\eta_{\mu\nu}^{(\pm)i} R_{\mu\nu} \tag{3.5.3.18}$$

$$\begin{aligned} F^{(-)1} &= R_{01} = -R_{23} = \frac{2m}{(r+m)^3}\left(e^0 \wedge e^1 - e^2 \wedge e^3\right) \\ F^{(-)2} &= R_{02} = -R_{31} = -\frac{m}{(r+m)^3}\left(e^0 \wedge e^2 - e^3 \wedge e^1\right) \\ F^{(-)3} &= R_{03} = -R_{12} = \frac{m}{(r+m)^3}\left(e^0 \wedge e^3 - e^1 \wedge e^2\right) \end{aligned} \tag{3.5.3.19}$$

where it is obvious that due to the absence of self-dual curvature, there are no $SU(2)_+$ gauge fields, ie. $F^{(+)i} = 0$ and thus field strengths are anti-self dual ($F = - * F$) which of course, coincide with the curvature tensor (3.5.3.15). In terms of 2-forms, the independent components are given by :

$$R^{(-)}_{0101} = R^{(-)}_{2323} = -R^{(-)}_{0123} = \frac{2m}{(r+m)^3} \tag{3.5.3.20}$$

$$R^{(-)}_{0202} = R^{(-)}_{1313} = R^{(-)}_{0213} = -\frac{m}{(r+m)^3} \tag{3.5.3.21}$$

$$R^{(-)}_{0303} = R^{(-)}_{1212} = -R^{(-)}_{0213} = -\frac{m}{(r+m)^3} \tag{3.5.3.22}$$

This lets us compute the Ricci tensors and scalar in accordance with the formula:

$$\mathcal{R}_{ik} = g^{jl}\mathcal{R}_{ijkl} = \delta^{jl}\mathcal{R}_{ijkl} \qquad R = \delta^{ik}\mathcal{R}_{ik} \tag{3.5.3.23}$$

$$\therefore \quad \mathcal{R}_{00} = \mathcal{R}_{11} = \mathcal{R}_{22} = \mathcal{R}_{33} = 0 \qquad\qquad R = 0 \tag{3.5.3.24}$$

Since Ricci tensors vanish, Taub-NUT is clearly a vacuum solution of Einstein's equations.

Topological Invariants

Topological invariants are analogous to an overall charge distributed in the manifold. In the gravity side, there are two topological invariants associated with the Atiyah-Patodi-Singer index theorem for a four dimensional elliptic complex [88, 154]: the Euler characteristic $\chi(M)$ and the Hirzebruch signature $\tau(M)$, which can be expressed as integrals of four-manifold curvature.

Recall that in electromagnetic theory, the field action is given by:

$$S = -\frac{1}{16\pi}\int d\Omega \; F_{ij}F^{ij} = -\frac{1}{16\pi}\int F\wedge F$$

$$\text{where} \qquad F = \frac{1}{2}F_{ij}dx^i\wedge dx^j \qquad \text{and} \qquad \varepsilon^{ijkl}d\Omega = dx^i\wedge dx^j\wedge dx^k\wedge dx^l$$

The equations of motion are obtained by solving for minimum variation of the electromagnetic field action. We merely apply these equations to compute topological invariants as integrals analogous to action. We can write for the general Lagrangian:

$$\begin{aligned}\mathcal{L} &= c^{abcd}R_{ab}\wedge R_{cd} = c^{abcd}F^{(\pm)m}_{ab}F^{(\pm)n}_{cd}\eta^{(\pm)m}_{ij}\eta^{(\pm)n}_{kl}\varepsilon^{ijkl}d\Omega \\ &= \pm 2d\Omega c^{abcd}F^{(\pm)m}_{ab}\partial_c A^{(\pm)m}_d \qquad\qquad \left(\text{where} \quad \varepsilon^{ijkl}d\Omega = e^i\wedge e^j\wedge e^k\wedge e^l\right)\end{aligned} \tag{3.5.3.25}$$

Applying Lagrange's equation gives the contracted Bianchi identity for curvature as:

$$\partial_c\left(\frac{\partial\mathcal{L}}{\partial(\partial_c A^{(\pm)m}_d)}\right) = \pm 2c^{abcd}\partial_c F^{(\pm)m}_{ab} = 0 \tag{3.5.3.26}$$

Conversely, we can say that Bianchi identity for $SU(2)_\pm$ gauge fields is at the root of the invariance of topological quantities. One can verify this starting from (3.5.3.26) and then working in reverse order to obtain the invariants.

Given that the boundary integral vanishes, the overall invariant is computed only from the bulk part. For non-compact manifolds like Taub-NUT, there are additional boundary terms neither separated into self-dual nor anti-self-dual parts unlike the volume terms. They are the so-called eta-invariant $\eta_S(\partial M)$, given for k self-dual gravitational instantons by [155]

$$\eta_S(\partial M) = -\frac{2\epsilon}{3k} + \frac{(k-1)(k-2)}{3k} \qquad \begin{cases} \epsilon = 0; & \text{ALE boundary conditions} \\ \epsilon = 1; & \text{ALF boundary conditions} \end{cases} \tag{3.5.3.27}$$

Since Taub-NUT is a ALF hyperkähler four-manifold it has a non-vanishing eta-invariant which is equal to $-\frac{2}{3}$. According to calculations described in [156], upon applying curvature components of (3.5.3.15), the Euler characteristic χ and the Hirzebruch signature complex τ are:

$$\chi(M) = \frac{1}{32\pi^2}\int_M \varepsilon^{abcd} R_{ab} \wedge R_{cd} = 1 \tag{3.5.3.28}$$

$$\begin{aligned} \tau_{bulk}(M) &= -\frac{1}{12\pi^2}\left(\int_M R_{ab}\wedge R_{ab}\right)_{a<b} = \frac{2}{3} \\ \therefore \qquad \tau(M) &= \tau_{bulk}(M) + \eta_S(\partial M) = 0 \end{aligned} \tag{3.5.3.29}$$

One could say that the general form of various topological invariants can be written as:

$$\mathcal{C}(M) = \frac{1}{k\pi^2}\int_M c^{abcd} R_{ab}\wedge R_{cd} = \begin{cases} \frac{1}{k\pi^2}\int_M F_{ab}\left(*F^{ab}\right); & c^{abcd} = \varepsilon^{abcd} \quad \text{(Euler Char.)} \\ \frac{1}{k\pi^2}\int_M F_{ab}F^{ab}; & c^{abcd} = g^{ac}g^{bd} \quad \text{(Hirzebruch Sign.)} \end{cases} \tag{3.5.3.30}$$

where c^{abcd} is contracting tensor defined in respect to the relevant circumstances.

3.5.4 Killing-Yano tensors and the Taub-NUT metric

There are tensors quadratic in momenta and conserved along geodesics, expressed as a vector $\boldsymbol{K}$ whose components transform among themselves under 3-dimensional rotations. They are very similar to the Runge-Lenz vector in the Kepler problem with components:

$$K^{(i)} = \frac{1}{2}K^{(i)\mu\nu}p_\mu p_\nu \tag{3.5.4.1}$$

Provided that $J^0 \neq 0$, such vectors usually satisfy the following property:

$$\boldsymbol{r}.\left(\boldsymbol{K} \pm \frac{H\boldsymbol{J}}{J^0}\right) = \frac{1}{2}\left(\boldsymbol{J}^2 - \left(J^0\right)^2\right) \tag{3.5.4.2}$$

where if $(J^0, \boldsymbol{J}, H, \boldsymbol{K})$ are all constant, the 3-dimensional position vector $\boldsymbol{r}$ lies in a plane. Using (3.5.4.2) and the relation $J^0 = \dfrac{\boldsymbol{r}.\boldsymbol{J}}{r}$, we can see that:

$$\boldsymbol{r}.\boldsymbol{K} = \mp rH + \frac{1}{2}\left(\boldsymbol{J}^2 - \left(J^0\right)^2\right) \tag{3.5.4.3}$$

In Taub-NUT geometry, there are also 4 completely antisymmetric Killing tensors known as Killing-Yano (KY) tensors. Three of these are complex structures, realizing quaternionic algebra since the Taub-NUT manifold is hyper-Kähler. The fourth is a scalar with a non-vanishing field strength and it exists by virtue of the metric being of Petrov type D. Their existence is implied by a triplet of symmetric 2nd rank Killing tensors called the Stäckel-Killing tensor satisfying:

$$D_{(\lambda}K^{(i)}_{\mu\nu)} = 0 \tag{3.5.4.4}$$

We will examine properties of KY tensors relevant for studying Taub-NUT symmetries. Before that, let us list some references that initiated the study of such dynamical symmetries.

Dynamical symmetries of the Kaluza Klein monopole were discussed in detail by Feher in [157]. The dynamics of two non-relativistic BPS monopoles was described using Atiyah-Hitchin metric (Taub-NUT being a special case), the corresponding $O(4)/O(3,1)$ symmetry discovered in [158], and applied to calculate the underlined motion group-theoretically in [159]. The symmetry was then extended to $O(4,2)$ in [160] and [161]. In [160] Gibbons et. al discussed dynamical symmetries of multi-centre metrics and applied the results to the scattering of BPS monopoles and fluctuations around them, giving a detailed account of the hidden symmetries of the Taub-NUT. The hidden symmetries in large-distance interactions between BPS monopoles and of the fluctuations around them are traced to the existence of a KY tensor on the self-dual Taub-NUT. The global action on classical phase space of these symmetries was discussed in [162] and the quantum picture involving the "dynamical groups" $SO(4)$, $SO(4,1)$ and $SO(4,2)$ was also given. A comprehensive review of the dynamical symmetry can be found in [163]. Supersymmetry and extension to spin has also been studied in [164, 165].

Yano and Stäckel tensors

We can construct these KY tensors in terms of simpler objects known as Yano tensors that are antisymmetric rank 2 tensors satisfying the Killing like equation. Thus, the covariant derivative is antisymmetric over permutations of all possible pairs of indices. This allows us to write the covariant derivative of the Yano tensor in terms of the cyclic permutations as:

$$f_{\mu\nu} = -f_{\nu\mu} \qquad \nabla_\mu f_{\nu\lambda} + \nabla_\nu f_{\mu\lambda} = 0 \tag{3.5.4.5}$$

$$\nabla_\mu f_{\nu\lambda} = \nabla_\nu f_{\lambda\mu} = \nabla_\lambda f_{\mu\nu} = \nabla_{[\mu} f_{\nu\lambda]} \quad = \quad \frac{1}{3}\left(\nabla_\mu f_{\nu\lambda} + \nabla_\nu f_{\lambda\mu} + \nabla_\lambda f_{\mu\nu}\right) \tag{3.5.4.6}$$

We can also construct symmetric Killing tensors of rank 2 by symmetrized multiplication:

$$K^{(ab)}_{\mu\nu} = \frac{1}{2}\left(f^{(a)\lambda}_{\mu} f^{(b)}_{\lambda\nu} + f^{(b)\lambda}_{\mu} f^{(a)}_{\lambda\nu}\right) \quad \equiv \quad \frac{1}{2}\left(f^{(a)\lambda}_{\mu} f^{(b)}_{\lambda\nu} + f^{(a)\lambda}_{\nu} f^{(b)}_{\lambda\mu}\right) = K^{ab}_{(\mu\nu)} \tag{3.5.4.7}$$

These symmetric Killing tensors satisfy the condition (3.5.4.4). The Taub-NUT manifold admits 4 such KY tensors, given by a scalar f^0 and three components that transform as a vector f^i $\forall$ $i = 1, 2, 3$. We can form triplets of symmetric Killing tensors as in (3.5.4.7), given by setting $a = 0$ and $b = i$:

$$K^{(i)}_{\mu\nu} = K^{(0i)}_{\mu\nu} = \frac{1}{2}\left(f^{0\lambda}_{\mu} f^{i}_{\lambda\nu} + f^{i\lambda}_{\mu} f^{0}_{\lambda\nu}\right) \qquad i = 1, 2, 3 \tag{3.5.4.8}$$

Using (3.5.4.5) we can see how they obey (3.5.4.4) as follows:

$$\nabla_\gamma K^{ij}_{(\mu\nu)} + \nabla_\mu K^{ij}_{(\nu\gamma)} + \nabla_\nu K^{ij}_{(\gamma\mu)} = 0,$$

$$\nabla_{(\gamma} K^{ij}_{\mu\nu)} = 0 \quad \Rightarrow \quad \nabla_{(\gamma} K^{i}_{\mu\nu)} \equiv \nabla_{(\gamma} K^{0i}_{\mu\nu)} = 0 \tag{3.5.4.9}$$

This allows us to construct the tensors of (3.5.4.1) that are quadratic in momenta, showing how to get Stäckel tensors from the KY tensors. However, since the KY tensor is anti-symmetric, it cannot be used to form polynomials with components of the same vector. Thus, it will have to be a mixed product of components of different vector quantities, as found in case of the angular momentum, a product between one position and one momentum component each. Applying Holten's algorithm yields the Killing equation in (3.5.4.5).

Euclidean Taub-NUT

The Taub-NUT metric [166] admits four such Yano tensors written as the following 2-forms:

$$f^0 = 4\left(d\psi + \cos\theta\ d\phi\right) \wedge dr + 2r\left(r \pm 1\right)\left(r \pm 2\right)\sin\theta\ d\theta \wedge d\phi \tag{3.5.4.10}$$

$$f^i = \pm 4\left(d\psi + \cos\theta\ d\phi\right) \wedge dx^i - \varepsilon^i{}_{jk} f(r)\ dx^j \wedge dx^k, \qquad \forall i, j, k = 1, 2, 3 \tag{3.5.4.11}$$

One can always find Killing tensors embedded within conserved quantities, as evident from the Poisson Brackets of any conserved quantity expanded ala Holten's algorithm. The coefficient from Laplace-Runge-Lenz vector is analogous to the Killing-Stäckel tensor K_{ij}, so we can argue:

$$Q^{(2)} = K_{ij}\Pi^i\Pi^j \equiv \frac{1}{2}C^{(2)}_{ij}\Pi^i\Pi^j \tag{3.5.4.12}$$

Now the angular momentum co-efficients according to (3.5.1.29) are:

$$C^{(0)} = q\ g_{jk}(\vec{x})\ \frac{x^j}{r}\theta^k \qquad C^{(1)}_i = -g_{im}(\vec{x})\varepsilon^m{}_{jk}\theta^k x^j \tag{3.5.4.13}$$

If we write $C^{(1)}_i = f_{ik}\theta^k$ (see Appendix 6.2), using Holten's Algorithm gives:

$$\nabla_j C^{(1)}_i = \nabla_j f_{ik}\theta^k = -g_{im}(\vec{x})\varepsilon^m{}_{jk}\theta^k$$
$$\nabla_i C^{(1)}_j + \nabla_j C^{(1)}_i = 0 \qquad \Rightarrow \qquad \left(\nabla_i f_{jk} + \nabla_j f_{ik}\right)\theta^k = 0$$

which is the Killing equation (3.5.4.5). Thus, we can say that the KY tensor is

$$f^0_{jk} = g_{jk}(\vec{x}) \qquad \Rightarrow \qquad f^j_{0k} = \delta^j_k \tag{3.5.4.14}$$

$$f^i_{jk} = \varepsilon^i{}_{jk} \qquad \Rightarrow \qquad f^i = \varepsilon^i{}_{jk} e^j \wedge e^k \tag{3.5.4.15}$$

such that the square of it gives the Stäckel tensor

$$K^k_{ij} = f^0{}_{im} f^{km}{}_j \tag{3.5.4.16}$$

This shows how Killing tensors are embedded within the conserved quantities. We can choose four combinations of three indices out of the available four. Since Taub-NUT can be written in an alternate form given by (3.5.1.10), the vierbeins of the metric are given by:

$$e^0 = \frac{4\left(d\psi + \vec{A}.d\vec{x}\right)}{\sqrt{f(r)}} \qquad e^i = \sqrt{f(r)}\ dx^i \tag{3.5.4.17}$$

So, according to our theory, we should have

$$\begin{aligned} f^i &= -\varepsilon^i{}_{jk}\ e^j \wedge e^k + \delta^i{}_k\ e^0 \wedge e^k \\ &= -\varepsilon^i{}_{jk} f(r) dx^j \wedge dx^k \pm 4\left(d\psi + \vec{A}.d\vec{x}\right) \wedge dx^i \end{aligned} \tag{3.5.4.18}$$

This result so far is comparable with the result (3.5.4.11), so we have a possible method for constructing Killing-Yano tensors from the coefficients of conserved quantities. Their covariant exterior derivatives and their properties are given by:

$$Df^0 = \nabla_\gamma f^0_{\mu\nu} dx^\gamma \wedge dx^\mu \wedge dx^\nu = r\,(r \pm 2)\sin\theta dr \wedge d\theta \wedge d\phi, \tag{3.5.4.19}$$

$$Df^i = 0, \qquad \forall i = 1,2,3. \tag{3.5.4.20}$$

From the results above, we can infer that the covariant derivatives hold following properties:

$$\nabla_\gamma f^0_{\mu\nu} = \nabla_\mu f^0_{\nu\gamma} = \nabla_\gamma \nu f^0_{\gamma\mu}, \qquad \nabla_\gamma f^i_{\mu\nu} = 0 \qquad i = 1,2,3. \tag{3.5.4.21}$$

showing that they obey the condition for covariant derivatives of KY tensors. As shown in (3.5.4.8), these tensors can form a symmetric triplet or a vector of Killing tensors. They also exhibit the mutual anti-commutation property:

$$f^i f^j = -\delta^{ij} + \varepsilon^{ij}{}_k f^k, \qquad \begin{cases} \{f^i, f^j\} = f^i f^j + f^j f^i = -2\delta^{ij}, \\ [f^i, f^j] = f^i f^j - f^j f^i = 2\varepsilon^{ij}{}_k f^k. \end{cases} \tag{3.5.4.22}$$

proving that they are complex structures realizing the quaternion algebra. This implies that the 2-forms f^i are objects in the quaternionic geometry and possibly **hyperkähler** structures. This leads us to the next subsection where we examine the hyperkähler structure of the Taub-NUT.

Graded Lie-algebra via Schouten-Nijenhuis Brackets

We will now see if the KY tensors of the Taub-NUT metric exhibit Lie algebra under the action of Schouten-Nijenhuis Brackets. If they do, it would allow us to form higher order KY tensors from lower order ones of rank greater than 1. In particular it is noteworthy in this context that, Kastor et. al already found that KY tensors on constant curvature space-times do form Lie algebras with respect to the SN bracket [167].

The Schouten-Nijenhuis Bracket (SNB) is a bracket operation between multivector fields. The SNB for two such fields $A = A^{i_1 i_2 \dots i_m} \bigwedge_{k=1}^m \partial_{i_k}$; $B = B^{j_1 j_2 \dots j_n} \bigwedge_{k=1}^n \partial_{j_k}$, is given by

$$\begin{aligned} C^{a_1 \dots a_{m+n-1}} &= [A, B]^{a_1 \dots a_{m+n-1}}_{SN} \\ &= m A^{c[a_1 \dots a_{m-1}} \nabla_c B^{a_m \dots a_{m+n-1}]} + n\,(-1)^{mn} B^{c[a_1 \dots a_{n-1}} \nabla_c A^{a_n \dots a_{m+n-1}]}. \end{aligned} \tag{3.5.4.23}$$

This new tensor is completely antisymmetric, fulfilling the first requirement to be considered a KY tensor. All that remains is for its covariant derivative to exhibit the same Killing equation (3.5.4.6) relevant to such tensors. Now, we will use an important identity (see (6.4.3) in Appendix 6.4) for KY tensors:

$$\therefore \qquad \nabla_a \nabla_b K_{c_1 c_2 \dots c_n} = (-1)^{n+1} \frac{n+1}{2} R_{[bc_1|a|}{}^d K_{c_2 c_3 \dots c_n]d}. \tag{3.5.4.24}$$

we get upon applying to the covariant derivative of this new tensor:

$$\begin{aligned} \nabla_b C_{a_1 \dots a_{m+n-1}} = &-(m+n)\left(\nabla_c A_{[ba_1 \dots a_{m-1}}\right) \nabla^c B_{a_m \dots a_{m+n-1}]} \\ &- (m+n) A^c{}_{[a_1 \dots a_{m-1}} R_{|bd|ca_m} B_{a_{m+1} \dots a_{m+n-1}]}{}^d. \end{aligned} \tag{3.5.4.25}$$

The 1st term easily shows anti-symmetry of index b with other indices, but the 2nd term exhibits it only under certain circumstances. One could say that by symmetry properties of the curvature tensor, in maximally symmetric spaces it could be expressed as:

$$R_{abcd}(x) = f(x) g_{ij}(x) \varepsilon^{i}{}_{ab} \varepsilon^{j}{}_{cd} = f(x) \left\{ g_{ac}(x) g_{bd}(x) - g_{ad}(x) g_{bc}(x) \right\}. \tag{3.5.4.26}$$

So, for cases of constant curvature $f(x) = k$, we could write

$$(R_{abcd})_{const} = k \left\{ g_{ac}(x) g_{bd}(x) - g_{ad}(x) g_{bc}(x) \right\}. \tag{3.5.4.27}$$

Thus, upon applying the constant curvature formula of (3.5.4.27) to (3.5.4.25), we will get

$$\begin{aligned} \nabla_b C_{a_1 \dots a_{m+n-1}} = -(m+n) \Big[& \left(\nabla_c A_{[b a_1 \dots a_{m-1}} \right) \nabla^c B_{a_m \dots a_{m+n-1}]} \\ & - k A_{[a_1 \dots a_{m-1}} B_{a_m \dots a_{m+n-1} b]} \Big] = \nabla_{[b} C_{a_1 \dots a_{m+n-1}]}. \end{aligned} \tag{3.5.4.28}$$

Clearly this matches the property eq.(3.5.4.6) expresses, showing that it is also a KY tensor. So the SNB of any two KY tensors in spaces of constant curvature is also a KY tensor.

However, as evident from (3.5.3.15), the curvature of the Taub-NUT metric is not constant, allowing us to conclude that its KY tensors do not exhibit Lie algebra under SN Brackets. Thus, we cannot produce higher order KY tensors using the lower order ones for the Taub-NUT as shown in [168]. So, we are limited to the set of four available rank two KY tensors.

3.5.5 Hyperkähler structure and the KY tensors

Now we will consider the hyperkähler structures related to the Taub-NUT metric. To begin with, we will define both, kähler and hyperkähler structures.

Definition 3.5.1. Kähler manifold: If a complex manifold M has a hermitian metric g and a fundamental 2-form ω which is closed ($d\omega = 0$) then M is a Kähler manifold and ω is a Kähler form.

The connection between the metric g and the Kähler form ω is:

$$\omega_{\mu\nu} = J_{\mu}{}^{\lambda}.g_{\lambda\nu} = (Jg)_{\mu\nu}, \tag{3.5.5.1}$$

where J is the complex structure, for which $J^2 = -1$.

Definition 3.5.2. Hyperkähler manifold: If M is a hyper-complex manifold with a hyper-Hermitian metric g and a triplet of fundamental forms $\vec{\omega}$ which are closed ($d\vec{\omega} = 0$) then M is a Hyperkähler manifold. It is the same as the Kähler manifold except that there are more than one type of complex structures. In case of 4 dimensions, there are 3 such integrable complex structures (i, j, k), and they obey the algebraic relations:

$$i^2 = j^2 = k^2 = ijk = -1. \tag{3.5.5.2}$$

This would also imply that there are corresponding number of different 2-forms available in this case, known as the Hyperkähler forms, given by:

$$\omega^{i}_{\mu\nu} = J^{i}_{\mu}{}^{\lambda}.g_{\lambda\nu} = \left(J^i g\right)_{\mu\nu}, \tag{3.5.5.3}$$

where $g_{\lambda\nu}$ is the hyper-hermitian metric and $J^i_{\mu\lambda}$ is the almost complex structure exhibiting quaternion algebra

$$J_\alpha J_\beta = -\delta_{\alpha\beta} I + \varepsilon_{\alpha\beta}{}^{\gamma} J_\gamma. \tag{3.5.5.4}$$

Thus, we can see that the hyperkähler structures exhibit the same algebra:

$$\begin{aligned} \left(J^i J^j\right)_{\mu\nu} &= J^i_{\mu\rho}\, g^{\rho\sigma} J^j_{\sigma\nu} \qquad \left[J^i, J^j\right]_{\mu\nu} = 2\varepsilon^{ij}{}_k J^k_{\mu\nu}, \\ \left(\omega^i \omega^j\right)_{\mu\nu} &= \omega^i_{\mu\gamma}\omega^{j\gamma}{}_\nu = \left(J^{i\,\rho}_{\mu}.g_{\rho\gamma}\right) g^{\gamma\lambda}\left(J^{j\,\sigma}_{\lambda}.g_{\sigma\nu}\right) = J^{i\,\rho}_{\mu} J^{j\,\sigma}_{\rho} g_{\sigma\nu} = \left(J^i J^j g\right)_{\mu\nu}, \\ \therefore \qquad \left[\omega^i, \omega^j\right]_{\mu\nu} &= \left(\left[J^i, J^j\right] g\right)_{\mu\nu} = 2\left(\varepsilon^{ij}{}_k J^k g\right)_{\mu\nu} = 2\varepsilon^{ij}{}_k \omega^k_{\mu\nu}. \end{aligned}$$

These complex structures originate from the t'Hooft symbols which have 3 self dual and 3 anti-self dual components. That means we could have six different symplectic 2-forms. The almost complex structures J^i can be represented by t'Hooft symbols, which themselves can be given by linear combinations of antisymmetric tensor $\varepsilon^i{}_{jk}$ and delta function $\delta^i{}_j$.

$$J^i_{jk} = \varepsilon^i{}_{jk} \pm \frac{1}{2}\left(\delta^0{}_j \delta^i{}_k - \delta^0{}_k \delta^i{}_j\right). \tag{3.5.5.5}$$

Thus, we can argue that hyper-kähler structures given by (3.5.5.3) are:

$$\omega^i_{jk} = \left(J^i g\right)_{jk} = g_{jn}(\vec{x}) \left[\varepsilon^{in}{}_k \pm \frac{1}{2}\left(\delta^{0n}\delta^i{}_k - \delta^0{}_k \delta^{in}\right)\right]. \tag{3.5.5.6}$$

As introduced in (3.5.1.11) and following [169] we shall take a different form of the Taub-NUT

$$ds^2 = V(r)\ \delta_{ij}\ dx^i dx^j + V^{-1}(r)\left(d\tau + \vec{\sigma}.d\vec{r}\right)^2. \tag{3.5.5.7}$$

for which, the vierbeins, in a similar fashion to (3.5.4.17) are given by

$$e^0 = \frac{4\left(d\tau + \vec{\sigma}.d\vec{r}\right)}{\sqrt{V(r)}} \qquad e^i = \sqrt{V(r)}\ dx^i.$$

Thus, remembering that $g = \delta_{ij} e^i \otimes e^j$ the hyper-kähler forms are given by:

$$\omega^i = \omega^i_{jk} dx^j \wedge dx^k = J^i_{jk} e^j \wedge e^k \tag{3.5.5.8}$$

$$\begin{aligned} \omega^i = \left[\varepsilon^i{}_{jk} \pm \frac{1}{2}\left(\delta^0{}_j \delta^i{}_k - \delta^0{}_k \delta^i{}_j\right)\right] e^j \wedge e^k \quad &= \quad \varepsilon^i{}_{jk} V(r) dx^j \wedge dx^k - e^0 \wedge e^i, \\ \therefore \qquad \omega^i = \varepsilon^i{}_{jk} V(r) dx^j \wedge dx^k &\pm \left(d\tau \wedge dx^i + \sigma_n . dx^n \wedge dx^i\right) \end{aligned} \tag{3.5.5.9}$$

For the Taub-NUT, choosing only anti-self-dual components for $V(r) = l + \dfrac{1}{r}$ and restricting $\vec{\sigma}$ to lie on a plane ($\vec{\sigma} = (0, \sigma_2, \sigma_3)$), the reduced symplectic forms are:

$$\begin{aligned} \omega^1 &= dx^1 \wedge d\tau + \sigma_2 dx^1 \wedge dx^2 + \sigma_3 dx^1 \wedge dx^3 + \left(l + \frac{1}{r}\right) dx^2 \wedge dx^3 \\ \omega^2 &= dx^2 \wedge d\tau + \sigma_3 dx^2 \wedge dx^3 - \left(l + \frac{1}{r}\right) dx^1 \wedge dx^3 \\ \omega^3 &= dx^3 \wedge d\tau - \sigma_2 dx^2 \wedge dx^3 + \left(l + \frac{1}{r}\right) dx^1 \wedge dx^2 \end{aligned} \tag{3.5.5.10}$$

Table 3.2: Comparison between Killing-Yano Tensors and Hyperkähler Structures

i	**Killing-Yano tensor f^i**	**Hyperkähler structure ω^i**
i	$\pm 4\left(d\psi + A_n dx^n\right) \wedge dx^i - \varepsilon^i{}_{jk} f(r) dx^j \wedge dx^k$	$\pm\left(d\tau + \sigma_n . dx^n\right) \wedge dx^i + \varepsilon^i{}_{jk} V(r) dx^j \wedge dx^k$
1	$\mp 4 dx^1 \wedge \left(d\psi + A_n dx^n\right) + \left(1 + \frac{4}{r}\right) dx^2 \wedge dx^3$	$dx^1 \wedge \left(d\tau + \sigma_2 dx^2 + \sigma_3 dx^3\right) + \left(l + \frac{1}{r}\right) dx^2 \wedge dx^3$
2	$\mp 4 dx^2 \wedge \left(d\psi + A_n dx^n\right) - \left(1 + \frac{4}{r}\right) dx^1 \wedge dx^3$	$dx^2 \wedge \left(d\tau + \sigma_3 dx^3\right) - \left(l + \frac{1}{r}\right) dx^1 \wedge dx^3$
3	$\mp 4 dx^3 \wedge \left(d\psi + A_n dx^n\right) + \left(1 + \frac{4}{r}\right) dx^1 \wedge dx^2$	$dx^3 \wedge \left(d\tau - \sigma_2 dx^3\right) + \left(l + \frac{1}{r}\right) dx^1 \wedge dx^2$

This construction of hyperkähler structures is similar to how spatial KY tensors were deduced, proving that the KY tensors are the hyperkähler structures of the Taub-NUT metric.

Few points are worth mentioning here. By studying the G_2 holonomy equation for biaxial anti-self dual Bianchi IX base Gibbons et.al [170] found that the associated first order equations satisfied by the metric coefficients yield the self-dual Ricci flat Taub-NUT metrics where $SO(3) \subset U(2)$ rotates the three hyperkähler forms as a triplet.

Chapter 4

Relativistic Mechanics of accelerating particles

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4.1 Introduction

In relativistic mechanics, we describe a geometric perspective of dynamics. This means that we start by describing pseudo-Riemannian spaces via metrics (M, g) by which we shall measure infinitesimal arc lengths in such spaces. Dynamical trajectories or geodesics between any two chosen fixed points are the shortest path in terms of integrated length in between. We are familiar with the usage of infinitesimal arc length in special relativity for flat spaces.

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu = c^2 dt^2 - |d\boldsymbol{x}|^2. \tag{4.1.1}$$

In general or curved spaces, the general infinitesimal arc length element is given by:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{00} c^2 dt^2 + 2g_{0i} c\, dt\, dx^i + g_{ij} dx^i dx^j, \qquad i, j = 1, 2, 3. \tag{4.1.2}$$

which becomes flat when $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. Thus, we must derive mechanical formulation from (4.1.2) to correctly describe relativistic mechanics in general on curved spaces. space-time for the usual problems dealt with in classical mechanics simply involve a 4-potential $A^\mu = (U, -\boldsymbol{A})$, for which, we shall have the spatial terms of the metric are flat $(g_{ij} = -\delta_{ij})$.

One important rule for a metric to abide by is that it must be invariant under the Lorentz transformation, which is easily formulated in special relativity for free particles travelling at constant velocity. However, the familiar Lorentz transformation does not preserve metrics describing trajectories for particles in the presence of potential fields. Thus we must define the Lorentz transformation in such a way that it locally preserves such metrics, meaning that the metric at a point is invariant only under the transformation rule defined at that same point. Its local nature means that position co-ordinates cannot be transformed as in special relativity.

There is an alternate ad hoc approach taken to formulate the relativistic Lagrangian, employed in publications by Harvey [171] and Babusci et al [172], to describe the dynamics of the relativistic oscillator and Kepler. It essentially draws from the conventional formulation of the Lagrangian provided by Goldstein in [173]

$$L = -mc^2\sqrt{1-\left(\frac{v}{c}\right)^2} - U. \tag{4.1.3}$$

where the kinetic energy term is kept separate from the potential term. Although (4.1.3) gives the correct relativistic answers for practical problems, this Lagrangian is not Lorentz covariant. One may suspect that under some approximations, the formulation born from the metric will transform into the conventional Lagrangian based approach.

The conventional gravitational field yields conserved dynamics in a central force field without drag. However, motion involving drag is a significant topic in the study of dynamical systems, describing realistic situations, with practical applications mostly in engineering. It would naturally be very interesting to see what kinds of space-times and gravitational fields produce dynamics involving drag. Such systems are not always integrable unless certain conditions are fulfilled by the drag co-efficient functions and the force-field functions. The relativistic generalizations of Lagrangian/Hamiltonian systems with position-dependent mass [174, 175, 176] could be treated within this formalism. Such systems are in some cases equivalent to constant mass motion on curved spaces, and some nonlinear oscillators can be interpreted in this setting.

This chapter is organized as follows: Section 4.2 is devoted to the preliminaries on the formulation of special relativistic mechanics in flat space-time, followed by static curved spaces dealt with in classical mechanics. There we deduce the relativistic deformation of the Euler-Lagrange equation, and a conserved quantity related to such mechanics. We also describe relativistic Hamiltonian mechanics in curved spaces.

Section 4.3 deals with the modification of the Lorentz transformation under which such metrics are invariant. This is necessary since the regular Lorentz transformation, designed to work for free particles in the case of special relativity, will not suffice for particles accelerating under the influence of a potential field. Then, we will deduce the formulas for time-dilation, length contraction, and gravitational redshift from the modified Lorentz transformation formula.

Section 4.4 will list the various approximations that can be made and how they affect our formulations. Here we will verify if the relativistic Lagrangian under any of these approximations transforms into the conventional one employed by Harvey and Babusci et al.

Section 4.5 will cover formulation of the relativistic 2D oscillator using our approach. Here we will verify if the Bohlin-Arnold-Vasiliev duality between relativistic Kepler and Hooke systems holds in such non-classical settings, and what approximations, if any, are required. Such dualities are observable in Bertrand space-time metrics which correspond either to oscillator or to Kepler systems on the associated three dimensional curved spatial manifold [177], which could be studied from this perspective.

Section 4.6, finally, will study relativistic damped mechanical systems and redefine the Chiellini integrability condition in relativistic form. Then we will deduce the related metric for damped systems and define its contact Hamiltonian structure.

4.2 Preliminary: Relativistic Mechanics

Here, we shall describe relativistic formulation first for free particle, then for classical static curved spaces with scalar potentials. The general space-time metric for curved spaces in the locality of potential sources can be formulated by defining (4.1.2) as a perturbation around the flat space-time metric (4.1.1) that asymptotically vanishes towards infinity:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \qquad\qquad \lim_{x\to\infty} h_{\mu\nu}(x) = 0 \tag{4.2.1}$$

This is necessary to ensure that the metric is asymptotically flat at large distances from the potential field source.

$$\lim_{x\to\infty} ds^2 = \lim_{x\to\infty} g_{\mu\nu}(x)\ dx^\mu dx^\nu = \eta_{\mu\nu} dx^\mu dx^\nu$$

After that, we will discuss the Hamiltonian mechanical description. Then we shall explore various approximations and the modified results that follow.

4.2.1 Flat space

In special relativity, we are discussing free particle mechanics $U(\boldsymbol{x}) = 0$. This means that the metric describes flat space (4.1.1):

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = c^2 dt^2 - \left|d\boldsymbol{x}\right|^2$$

from which the Lagrangian derived is:

$$\widetilde{\mathcal{L}} = -mc\sqrt{\left(\frac{ds}{d\tau}\right)^2} = -mc\sqrt{c^2\dot{t}^2 - \left|\dot{\boldsymbol{x}}\right|^2} = -mc^2\dot{t}\sqrt{1-\left(\frac{|\boldsymbol{v}|}{c}\right)^2} \qquad \text{where} \quad \boldsymbol{v} = \frac{\dot{\boldsymbol{x}}}{\dot{t}}.$$

Alternatively, we can define the reparameterized Lagrangian $\mathcal{L}$ from the geometric action as:

$$S = \int_1^2 d\tau \widetilde{\mathcal{L}} = -mc^2 \int_1^2 dt\sqrt{1-\beta^2} = \int_1^2 dt\ \mathcal{L}, \qquad \text{where } \beta = \frac{|\boldsymbol{v}|}{c},$$

$$\Rightarrow \qquad \mathcal{L} = -mc\sqrt{\left(\frac{ds}{dt}\right)^2} = -mc^2\sqrt{1-\beta^2}.$$

This results in the relativistic momentum and energy given as:

$$p_\mu = \frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{x}^\mu} = \begin{cases} \boldsymbol{p} = \dfrac{\partial \widetilde{\mathcal{L}}}{\partial \dot{\boldsymbol{x}}} = \dfrac{m\boldsymbol{v}}{\sqrt{1-\beta^2}} = m\boldsymbol{v}\gamma \\ \mathcal{E} = -\dfrac{\partial \widetilde{\mathcal{L}}}{\partial \dot{t}} = \dfrac{mc^2}{\sqrt{1-\beta^2}} = mc^2\gamma = \dfrac{\boldsymbol{p}.\dot{\boldsymbol{x}} - \widetilde{\mathcal{L}}}{\dot{t}} = \boldsymbol{p}.\boldsymbol{v} - \mathcal{L} \end{cases}$$

where, we have designated a reparameterizing factor:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}. \tag{4.2.1.1}$$

from which, we have the familiar equations for relativistic energy:

$$\mathcal{E}^2 = \eta^{\mu\nu} p_\mu p_\nu = |p|^2 c^2 + m^2 c^4. \tag{4.2.1.2}$$

Furthermore, the singularity that occurs when $|\boldsymbol{v}| = c$ in the denominator in (4.2.1.1), ensures that the speed of light is never exceeded, establishing is as the physical upper limit of velocity in flat spaces as elaborated in [178, 179]. This concludes the basics of flat space. From here on, we will use the alternative convention involving mechanical systems parameterized wrt time. Next we shall look at curved spaces with scalar potentials.

4.2.2 Curved space for single scalar potential

When considering mechanics on classical static curved spaces, where the source of space-time curvature is one scalar potential source $U(\boldsymbol{x}) \neq 0$, the metric based arc length can also be used for curved spaces by including the potential $U(\boldsymbol{x})$ into the metric as a specific version of (4.1.2) following Gibbons' prescription [17, 7] as shown below:

$$ds^2 = g_{00}(\boldsymbol{x})c^2dt^2 - \left|d\boldsymbol{x}\right|^2 \qquad \text{where} \quad g_{00}(\boldsymbol{x}) = 1 + \frac{2U(\boldsymbol{x})}{mc^2}. \tag{4.2.2.1}$$

where the potential $U(\boldsymbol{x})$ is a factor in the only non-zero perturbative term $h_{00}(x)$ of the temporal metric term $g_{00}(\boldsymbol{x})$ that vanishes asymptotically, while $h_{ij}(\boldsymbol{x}) = 0 \ \forall \ i, j = 1, 2, 3$. The relativistic action S and Lagrangian $\mathcal{L}$ are:

$$S = \int_1^2 d\tau \ \mathcal{L} \qquad \mathcal{L} = -mc\sqrt{\left(\frac{ds}{d\tau}\right)^2} = -mc\sqrt{g_{00}(\boldsymbol{x})c^2\dot{t}^2 - |\dot{\boldsymbol{x}}|^2}.$$

If we include the potential linearly as a perturbation into the metric, then we have

$$\mathcal{L} = -mc^2\sqrt{\left(1 + \frac{2U(\boldsymbol{x})}{mc^2}\right)\dot{t}^2 - \left(\frac{|\dot{\boldsymbol{x}}|}{c}\right)^2} = -mc^2\dot{t}\sqrt{1 - \frac{2}{mc^2}\left(\frac{m|\boldsymbol{v}|^2}{2} - U(\boldsymbol{x})\right)}.$$

Now, we are familiar with the traditional non-relativistic or classical Lagrangian

$$L = T - U = \frac{m|\boldsymbol{v}|^2}{2} - U(\boldsymbol{x}), \qquad T = \frac{m|\boldsymbol{v}|^2}{2}. \tag{4.2.2.2}$$

Thus, the relativistic Lagrangian $\mathcal{L}$ is given by:

$$\mathcal{L} = -mc\sqrt{\left(\frac{ds}{d\tau}\right)^2} = -mc^2\sqrt{1 - \frac{2L}{mc^2}}. \tag{4.2.2.3}$$

Under the circumstances that we are dealing with a stationary free particle, we can define the ground-state relativistic Lagrangian as follows:

$$\left.\begin{matrix} \boldsymbol{v} = 0 \\ U(\boldsymbol{x}) = 0 \end{matrix}\right\} \quad \Rightarrow \quad L = 0 \quad \Rightarrow \quad \mathcal{L} = L_0 = -mc^2.$$

Thus, we can re-write the relativistic Lagrangian as follows:

$$\mathcal{L} = L_0\sqrt{1 + 2\frac{L}{L_0}}. \tag{4.2.2.4}$$

showing that the classical Lagrangian L is embedded within the relativistic Lagrangian $\mathcal{L}$. Furthermore, we can say that we recover flat space (4.2.1.2) when $g_{00}(\boldsymbol{x}) = 1$ ie. $(U(\boldsymbol{x}) = 0)$.

$$\frac{ds}{dt} = c\sqrt{1 + 2\frac{L}{L_0}} \quad \Rightarrow \quad \Gamma^{-1} = \frac{d\widetilde{t}}{dt} = \sqrt{1 + 2\frac{L}{L_0}} \quad \xrightarrow{U=0} \quad \gamma^{-1} = \sqrt{1 - \left(\frac{|\boldsymbol{v}|}{c}\right)^2}. \tag{4.2.2.5}$$

while the relativistic momenta are clearly Lorentz-covariant, with $\widetilde{t}$ being the proper time in the particle frame.

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \begin{cases} \boldsymbol{p} = \dfrac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}} = m\boldsymbol{v}\Gamma \\ \mathcal{E} = -\dfrac{\partial \mathcal{L}}{\partial \dot{t}} = mc^2 g_{00}(\boldsymbol{x})\Gamma = \dfrac{\boldsymbol{p}.\dot{\boldsymbol{x}} - \widetilde{\mathcal{L}}}{\dot{t}} \end{cases} \tag{4.2.2.6}$$

From the momenta (4.2.2.6), we can see that the singularity that occurs when $\Gamma^{-1} = 0$ establishes a different physical upper limit for the velocity of a particle:

$$1 + 2\frac{L}{L_0} = 0 \qquad \Rightarrow \qquad g_{00}(\boldsymbol{x})c^2\dot{t}^2 - |\dot{\boldsymbol{x}}|^2 = 0 \qquad \Rightarrow \qquad |\boldsymbol{v}| = \left|\frac{d\boldsymbol{x}}{dt}\right| = c\sqrt{g_{00}(\boldsymbol{x})}.$$

which is less than the speed of light c since $g_{00}(\boldsymbol{x}) < 1$, ($U(\boldsymbol{x}) \leq 0$ for gravitational potentials). This deformation of the relativistic 4-momentum as a consequence of the change of the speed limit, is due to the change in length contraction and time- dilation due to the gravitational potential under a local Lorentz transformation.

Under the approximation $L << L_0$, binomially expanding (4.2.2.4) :

$$\mathcal{L} \xrightarrow{L<<mc^2=-L_0} L_0\left(1 + \frac{L}{L_0}\right) = L + L_0. \tag{4.2.2.7}$$

Thus, in the non-relativistic limit, L given by (4.2.2.2) will suffice to produce the equations of motion. One can alternatively say that the effective classical Lagrangian directly derived from the metric (4.2.2.1) is

$$\mathcal{L} \xrightarrow{L<<mc^2=-L_0} -mc^2\left(1 - \frac{L}{mc^2}\right) = L - mc^2 = -\frac{m}{2}\left(\frac{ds}{dt}\right)^2 - \frac{mc^2}{2},$$

$$L_{eff} = -\frac{m}{2}\left(\frac{ds}{dt}\right)^2 = \frac{m}{2}\left(|\dot{\boldsymbol{x}}|^2 - c^2 g_{00}(\boldsymbol{x})\right) \qquad g_{00}(\boldsymbol{x}) = 1 + \frac{2U(\boldsymbol{x})}{mc^2}. \tag{4.2.2.8}$$

We shall now proceed to analyze the relativistic equations of motion.

Relativistic equations of motion

Now we shall turn our attention to formulating of the equations of motion. The Euler-Lagrange equation is given by:

$$\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^i}\right) = \frac{\partial\mathcal{L}}{\partial x^i}.$$

when applied to (4.2.2.4), we get the relativistic equation of motion:

$$\frac{d}{dt}\left(m\Gamma\boldsymbol{v}\right) = -\Gamma\boldsymbol{\nabla}U(\boldsymbol{x}).\frac{d}{dt}\left(m\Gamma\boldsymbol{v}\right) = -\Gamma\boldsymbol{\nabla}U(\boldsymbol{x}). \tag{4.2.2.9}$$

then we can write using (4.2.2.5): $\quad \widetilde{\boldsymbol{v}} = \frac{d\boldsymbol{x}}{d\widetilde{t}} = \frac{dt}{d\widetilde{t}}\boldsymbol{v} = \Gamma\boldsymbol{v}$, making (4.2.2.9) into

$$m\frac{d^2\boldsymbol{x}}{d\widetilde{t}^2} = \Gamma\frac{d}{dt}\left(m\Gamma\boldsymbol{v}\right) = -\Gamma^2\boldsymbol{\nabla}U(\boldsymbol{x}). \tag{4.2.2.10}$$

If we expand Γ^2 in this equation, we will get:

$$\Gamma^2 = \left(1 + 2\frac{L}{L_0}\right)^{-1} = 1 - 2\frac{L}{L_0} + (-2)^2\left(\frac{L}{L_0}\right)^2 + ...$$

,

$$\therefore \qquad m\frac{d^2\boldsymbol{x}}{d\widetilde{t}^2} = -\boldsymbol{\nabla}U(\boldsymbol{x}) + 2\frac{L}{L_0}\boldsymbol{\nabla}U(\boldsymbol{x}) - .....$$

which is the equation of motion with additional terms associated with the force function. Alternatively, we can say that on applying the Euler-Lagrange equation to the relativistic Lagrangian (4.2.2.4), we will get:

$$\mathcal{L} = L_0\sqrt{1+2\frac{L}{L_0}}, \qquad \frac{\partial \mathcal{L}}{\partial v^i} = \frac{L_0}{2\mathcal{L}}\frac{\partial L}{\partial v^i} \qquad \frac{\partial \mathcal{L}}{\partial x^i} = \frac{L_0}{2\mathcal{L}}\frac{\partial L}{\partial x^i},$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v^i}\right) = \frac{L_0}{2\mathcal{L}}\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \left(\frac{\partial L}{\partial v^i}\right)\frac{L_0^3}{2\mathcal{L}^3}\frac{dL}{dt}.$$

Recalling (4.2.2.5), and writing $\Gamma = \dfrac{L_0}{\mathcal{L}}$ according to (4.2.2.4), we get from the Euler-Lagrange equation $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v^i}\right) - \frac{\partial \mathcal{L}}{\partial x^i} = 0$

$$\frac{L_0}{2\mathcal{L}}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial x^i}\right] - \left(\frac{\partial L}{\partial v^i}\right)\frac{L_0^3}{2\mathcal{L}^3}\frac{dL}{dt} = 0,$$

$$\therefore \qquad \left[\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial x^i}\right] = \Gamma^2\left(\frac{\partial L}{\partial v^i}\right)\frac{dL}{dt} \qquad \Gamma = \frac{L_0}{\mathcal{L}}. \tag{4.2.2.11}$$

Thus, we have a relativistic deformation (4.2.2.11) of the Euler-Lagrange equation for the classical Lagrangian L derived by applying the original Euler-Lagrange equation to the relativistic Lagrangian $\mathcal{L}$.

Under non-relativistic limits, $\Gamma \approx 1$, (4.2.2.10) becomes the more familiar form of the equation of motion given below that directly derive from Euler-Lagrange equations applied upon (4.2.2.2):

$$m\frac{d^2\boldsymbol{x}}{dt^2} = -\boldsymbol{\nabla}U(\boldsymbol{x}). \tag{4.2.2.12}$$

Similarly, the Euler-Lagrange equation that derives from the effective Lagrangian (4.2.2.8) is:

$$\ddot{\boldsymbol{x}} = -\frac{c^2}{2}\boldsymbol{\nabla}g_{00}(\boldsymbol{x}) \equiv -\frac{1}{m}\boldsymbol{\nabla}U(\boldsymbol{x}). \tag{4.2.2.13}$$

which is equivalent to (4.2.2.12). Under the circumstances that one has either the relativistic or classical equations of motion, there is an algorithm to deduce the space-time metric that generates such dynamics:

1. Convert the relativistic equation to the non-relativistic version (4.2.2.12) (with $\gamma = 1$).
2. Deduce $U(\boldsymbol{x})$ from the non-relativistic equation (4.2.2.12).
3. Use U to reproduce the space-time metric according to (4.2.2.8).

This algorithm allows us to formulate a curved space-time metric that reproduces the dynamics described by the equation of motion considered. If the force is not gravitational in nature, it essentially produces the equivalent curved space-time that can act as a gravitational substitute for the mechanical system that imitates its observed motion classically.

Conserved Quantity

We shall now regard a conserved quantity deduced from (4.2.2.10) comparable to the Hamiltonian and the geodesic equation of motion. The conserved quantity in question is:

$$\therefore \qquad K^2 = \frac{\left(1+\dfrac{2U}{mc^2}\right)^2}{1-\dfrac{2L}{mc^2}} = \left(\Gamma g_{00}(\boldsymbol{x})\right)^2. \tag{4.2.2.14}$$

where K is essentially a multiple of the relativistic energy given by (4.2.2.6). Applying (4.2.2.14) to (4.2.2.5), we can say that

$$\Gamma^2 = \left(\frac{dt}{d\widetilde{t}}\right)^2 = \frac{1}{1-\frac{2L}{mc^2}} = \frac{K^2}{\left(1+\frac{2U}{mc^2}\right)^2}.$$

Along a geodesic, the equation of motion in the particle frame (4.2.2.10) can be re-written as

$$m\frac{d^2\boldsymbol{x}}{d\widetilde{t}^2} = -\left(\frac{dt}{d\widetilde{t}}\right)^2 \boldsymbol{\nabla}U(\boldsymbol{x}) = -\frac{mc^2}{2}\frac{K^2}{\left(1+\frac{2U}{mc^2}\right)^2}\boldsymbol{\nabla}\left(1+\frac{2U}{mc^2}\right) = \frac{mc^2}{2}K^2\boldsymbol{\nabla}\left(1+\frac{2U}{mc^2}\right)^{-1},$$

$$\therefore \qquad \frac{d^2\boldsymbol{x}}{d\widetilde{t}^2} = \frac{K^2c^2}{2}\boldsymbol{\nabla}\left(g_{00}(\boldsymbol{x})\right)^{-1} = \frac{c^2}{2}\boldsymbol{\nabla}\left(\Gamma^2 g_{00}(\boldsymbol{x})\right). \tag{4.2.2.15}$$

which is one way of writing the relativistic equation of motion. If we are consider Lorentz transformations of space-time event intervals, and $g_{00} = g_{00}(\boldsymbol{x})$, we can say that

$$dt \longrightarrow d\widetilde{t} = \frac{d\widetilde{t}}{dt}dt \qquad \Rightarrow \qquad g_{00} \longrightarrow \widetilde{g}_{00} = \left(\frac{d\widetilde{t}}{dt}\right)^{-2} g_{00} = \Gamma^2 g_{00}(\boldsymbol{x}).$$

which lets us write the Lorentz-covariant equation of motion:

$$\frac{d^2\boldsymbol{x}}{d\widetilde{t}^2} = \frac{c^2}{2}\boldsymbol{\nabla}\widetilde{g_{00}}(\boldsymbol{x}). \tag{4.2.2.16}$$

Now we shall reproduce the known and familiar relativistic phenomena of time dilation and gravitational red-shift from this formulation as a test.

4.2.3 Relativistic Hamiltonian mechanics in curved spaces

The Hamiltonian formulation of classical mechanics is very useful, not just for its geometrical properties, but also for enabling extension of the classical theory into the quantum context via standard quantization. Having described a relativistic Lagrangian formulation for mechanics in the presence of a scalar potential, it is natural to also consider the Hamiltonian formulation.

Referring to (4.2.2.6), the relativistic energy for curved spaces is:

$$\mathcal{E} = \left[mc^2 + 2U(\boldsymbol{x})\right]\Gamma \qquad\qquad \mathcal{E}^2 = g_{00}(\boldsymbol{x})\left(|\boldsymbol{p}|^2c^2 + m^2c^4\right). \tag{4.2.3.1}$$

This would effectively make the Hamiltonian $\mathcal{H}$:

$$\mathcal{H} = \sqrt{g_{00}(\boldsymbol{x})}\sqrt{|p|^2c^2 + m^2c^4}.$$

and the Hamilton's equations of motion for $g_{00}(\boldsymbol{x}) = 1 + \dfrac{2U(\boldsymbol{x})}{mc^2}$:

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} = \sqrt{g_{00}(\boldsymbol{x})}\frac{c\boldsymbol{p}}{\sqrt{|p|^2 + m^2c^2}}, \\
\dot{\boldsymbol{p}} &= -\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}} = -\frac{\boldsymbol{\nabla} U(\boldsymbol{x})}{\sqrt{g_{00}(\boldsymbol{x})}}\sqrt{\left(\frac{|p|}{mc}\right)^2 + 1}.
\end{aligned}
\tag{4.2.3.2}
$$

Such formulation has been applied to study the relativistic version of the Quantum Harmonic Oscillator [180]. In the following section, we will elaborate on the Lorentz transformation operation for such metrics.

4.3 A modified Local Lorentz Transformation

An important issue that emerges is the invariance of such metrics under Lorentz transformations. The Lorentz transformation we are familiar with applies only to special relativity, where we deal with free particles.

The conventional Lorentz boost of co-ordinates x of frame F to $\widetilde{x}$ of frame $\widetilde{F}$ is:

$$
\begin{aligned}
c\widetilde{t} &= \gamma ct - \gamma\beta x \\
\widetilde{x} &= \gamma\beta ct - \gamma x
\end{aligned}
\qquad
\begin{aligned}
&x = vt, \\
&\gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \beta = \frac{V_{\widetilde{F}F}}{c}.
\end{aligned}
\tag{4.3.1}
$$

where v is the constant velocity of the particle in frame F, and $V_{\widetilde{F}F}$ is the speed of frame $\widetilde{F}$ with respect to frame F. A better way to write (4.3.1) this locally is to replace: $x^\mu \to dx^\mu$ under which the space-time metric is invariant.

$$
\begin{aligned}
c\ d\widetilde{t} &= \gamma\ c\ dt - \gamma\beta\ dx \\
d\widetilde{x} &= \gamma\beta\ c\ dt - \gamma\ dx
\end{aligned},
\tag{4.3.2}
$$

$$
d\widetilde{s}^2 = c^2 d\widetilde{t}^2 - d\widetilde{x}^2 \quad = \quad ds^2 = c^2dt^2 - dx^2.
$$

The scenario we are dealing with in this case involves a particle under the influence of a potential field. The metric is easily invariant under rotations in the presence of spherically symmetric potentials, which leaves only boosts to be considered. Due to the presence of a potential, we are required to use a modified Lorentz boost operation, which we shall briefly derive here. The Lorentz boost equations are:

$$
\begin{aligned}
c\ d\widetilde{t} &= \Lambda^0_0\ c\ dt + \Lambda^0_1\ dx \\
d\widetilde{x} &= \Lambda^1_0\ c\ dt + \Lambda^1_1\ dx
\end{aligned}
\qquad
\Lambda = \begin{pmatrix} \Lambda^0_0 & \Lambda^0_1 \\ \Lambda^1_0 & \Lambda^1_1 \end{pmatrix}.
\tag{4.3.3}
$$

If we consider a Lorentz transformation to the particle frame, we should have $d\widetilde{x} = 0$, which means that from the second equation of (4.3.3), we have:

$$
\frac{\Lambda^1_0}{\Lambda^1_1} = \frac{v}{c} = \beta.
\tag{4.3.4}
$$

Furthermore, the determinant of the matrix Λ must be unity to preserve volume elements spanned by 4-vectors.

$$
\Lambda^0_0\Lambda^1_1 - \Lambda^0_1\Lambda^1_0 = 1.
\tag{4.3.5}
$$

Now, demanding that the metric be invariant under the transformation gives us another rule:

$$\Lambda^t G \Lambda = G \qquad \Rightarrow \qquad \Lambda^t = G\Lambda^{-1}G^{-1}, \qquad G = \begin{pmatrix} g_{00}(\boldsymbol{x}) & 0 \\ 0 & -1 \end{pmatrix}, \qquad g_{00}(\boldsymbol{x}) = 1 + \frac{2U(\boldsymbol{x})}{mc^2},$$

$$\begin{aligned} \Rightarrow \qquad \begin{pmatrix} \Lambda_0^0 & \Lambda_0^1 \\ \Lambda_1^0 & \Lambda_1^1 \end{pmatrix} &= \begin{pmatrix} g_{00}(\boldsymbol{x}) & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \Lambda_1^1 & -\Lambda_1^0 \\ -\Lambda_0^1 & \Lambda_0^0 \end{pmatrix} \begin{pmatrix} (g_{00}(\boldsymbol{x}))^{-1} & 0 \\ 0 & -1 \end{pmatrix}, \\ &= \begin{pmatrix} \Lambda_1^1 & g_{00}(\boldsymbol{x})\Lambda_1^0 \\ (g_{00}(\boldsymbol{x}))^{-1}\Lambda_0^1 & \Lambda_0^0 \end{pmatrix}, \\ \Rightarrow \qquad & \Lambda_0^0 = \Lambda_1^1, \qquad \Lambda_0^1 = g_{00}(\boldsymbol{x})\Lambda_1^0. \end{aligned} \tag{4.3.6}$$

Combining the equations (4.3.4) and (4.3.6) into (4.3.5) gives us:

$$\left(\Lambda_1^1\right)^2 - (g_{00}(\boldsymbol{x}))^{-1}\left(\Lambda_0^1\right)^2 = \left(\Lambda_1^1\right)^2\left(1 - (g_{00}(\boldsymbol{x}))^{-1}\beta^2\right) = 1,$$

$$\begin{aligned} \Lambda_0^0 = \Lambda_1^1 &= \frac{1}{\sqrt{1 - (g_{00}(\boldsymbol{x}))^{-1}\beta^2}} = \sqrt{g_{00}(\boldsymbol{x})}\Gamma, \\ \Lambda_0^1 &= g_{00}(\boldsymbol{x})\Lambda_1^0 = \sqrt{g_{00}(\boldsymbol{x})}\beta\Gamma. \end{aligned} \tag{4.3.7}$$

Thus, using (4.3.7), the modified Lorentz boost matrix (4.3.3) is given by:

$$\Lambda = \begin{pmatrix} \Gamma\sqrt{g_{00}} & -\beta\Gamma\left(\sqrt{g_{00}}\right)^{-1} \\ -\beta\Gamma\sqrt{g_{00}} & \Gamma\sqrt{g_{00}} \end{pmatrix}, \tag{4.3.8}$$

$$d\tilde{s}^2 = \left(1 + \frac{2U(\boldsymbol{x})}{mc^2}\right)c^2 d\tilde{t}^2 - d\tilde{x}^2 \quad = \quad ds^2 = \left(1 + \frac{2U(\boldsymbol{x})}{mc^2}\right)c^2 dt^2 - dx^2.$$

Thus, we have a modified local Lorentz transformation that preserves the metric. For a $3+1$ space-time, the modified local Lorentz boost matrix between one co-ordinate and time would be written as:

$$\Lambda = \begin{pmatrix} \Gamma\sqrt{g_{00}} & -\beta\Gamma\left(\sqrt{g_{00}}\right)^{-1} & 0 & 0 \\ -\beta\Gamma\sqrt{g_{00}} & \Gamma\sqrt{g_{00}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.3.9}$$

The local nature of this transformation should not be surprising since according to the Equivalence principle, at any point on a curved manifold, there always exists a diffeomorphism that transforms it locally into a flat manifold. This means that alternatively, within the locality of that point in the local inertial frame, we can transform the problem into special relativity and apply the regular Lorentz transformation (4.3.2).

Naturally, it is not possible to perform a global co-ordinate transformation like (4.3.1). In fact, we must understand that (4.3.1) derives from (4.3.2) via integration, and not the other way around via differentiation. This is because special relativity, as the name implies, describes a special case where space-time is isotropic due to the absence of any potentials and any global co-ordinates are described as $x^\mu = \int_1^2 d\tau\, \dot{x}^\mu = \dot{x}^\mu\tau$.

4.3.1 Time-dilation and length contraction

One of the reasons a speed limit exists is the Minkowskian signature of space-time, which allows null geodesics for non-null space-time intervals. This results in phenomena such as time-dilation and length contraction. The inclusion of a gravitational potential results in a modification of such phenomena, just as it has resulted in modification of the Lorentz transformation.

Since we are discussing a modified local Lorentz transformation (4.3.9), let us define the local co-ordinates in the neighbourhood of x where g_{00} is roughly constant:

$$x \longrightarrow x + \chi, \qquad t \longrightarrow t + \tau. \tag{4.3.1.1}$$

Now consider a problem where in a frame S, a stationary particle lies at position $x = l$ along the x-axis, and S moves at a velocity $v = \beta c$ along the x-axis wrt an observer in frame $\widetilde{S}$.

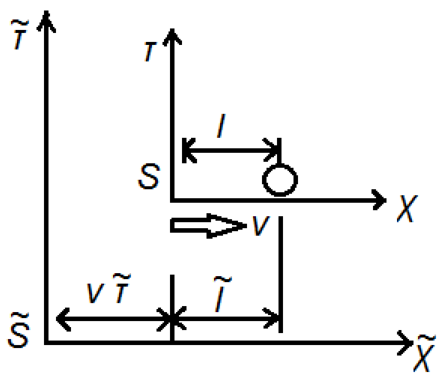

Figure 4.1: The Lorentz frames for time dilation and length contraction

Using (4.3.1.1) the local Lorentz transformation equations from S to $\widetilde{S}$ are:

$$\begin{aligned} c\, d\widetilde{\tau} &= \Gamma\sqrt{g_{00}(x)}\; c\, d\tau + \beta\Gamma\left(\sqrt{g_{00}(x)}\right)^{-1}\, d\chi \\ d\widetilde{\chi} &= \beta\Gamma\sqrt{g_{00}(x)}\; c\, d\tau + \Gamma\sqrt{g_{00}(x)}\; d\chi \end{aligned} \tag{4.3.1.2}$$

Since the particle is stationary in S ($d\chi = 0$), the first equation of (4.3.1.2) gives us:

$$d\widetilde{\tau} = \Gamma\sqrt{g_{00}(x)}d\tau \qquad \Rightarrow \qquad \boxed{d\widetilde{\tau} = \Gamma\sqrt{g_{00}(x)}d\tau}. \tag{4.3.1.3}$$

Integration of (4.3.1.2) including constants of integration yields:

$$\begin{aligned} c\widetilde{\tau} &= \Gamma\sqrt{g_{00}(x)}\; c\left(\int d\tau + \tau_0\right) + \beta\Gamma\left(\sqrt{g_{00}(x)}\right)^{-1}\left(\int d\chi + l\right) \\ \widetilde{\chi} &= \beta\Gamma\sqrt{g_{00}(x)}\; c\left(\int d\tau + \tau_0\right) + \Gamma\sqrt{g_{00}(x)}\left(\int d\chi + l\right) \end{aligned} \tag{4.3.1.4}$$

Since $\int d\chi = 0$ in the particle frame S, we set the following boundary conditions

$$\int d\tau = 0 \quad \Rightarrow \quad \widetilde{\tau} = \int d\widetilde{\tau} = 0 \qquad \Rightarrow \qquad c\tau_0 = -\beta\left(g_{00}(x)\right)^{-1} l. \tag{4.3.1.5}$$

and using (4.3.1.5), the 2nd equation of (4.3.1.4) gives us:

$$\widetilde{\chi} = \beta\Gamma\sqrt{g_{00}(x)}\; c\int d\tau \;+\; \left(\sqrt{g_{00}(x)}\right)^{-1}\left(g_{00}(x)-\beta^2\right)\Gamma l,$$

Now since the particle is in motion in $\widetilde{S}$, it moves from its starting position $\widetilde{\chi}_0 = \widetilde{l}$ with the velocity of frame S wrt $\widetilde{S}$. Thus, using (4.3.1.3), the contracted length $\widetilde{l}$ is:

$$\widetilde{\chi} = \int d\widetilde{\chi} + \widetilde{\chi}_0 \;=\; \beta c\int d\widetilde{\tau} + \widetilde{l} \;\;=\;\; \beta c\Gamma\sqrt{g_{00}(x)}\int d\tau + \widetilde{l},$$

$$\Rightarrow\quad \widetilde{l} = \widetilde{\chi} - \beta c\int d\widetilde{\tau} = \left(\sqrt{g_{00}(x)}\right)^{-1}\left(g_{00}(x)-\beta^2\right)\Gamma l = \left(\sqrt{g_{00}(x)}\right)^{-1}\Gamma^{-1} l,$$

$$\boxed{\widetilde{l} = \left(\sqrt{g_{00}(x)}\right)^{-1}\Gamma^{-1} l.} \tag{4.3.1.6}$$

For flat space, we set $g_{00} = 1$, and $\Gamma \longrightarrow \gamma$, which should restore the original time-dilation and length-contraction rules of special relativity for a free particle.

Due to the lowered speed limit in comparison to the speed of light for special relativity, time intervals shall dilate and length intervals shall contract further in presence of a gravitational potential field.

4.3.2 Gravitational redshift

An alternative way to arrive at the time-dilation formula (4.3.1.3) is to write the metric in its two equivalent forms in the two frames $\widetilde{S}$ and S:

$$ds^2 = c^2 d\widetilde{\tau}^2\left[1 + 2\frac{L}{L_0}\right] = g_{00}\; c^2 d\tau^2 \qquad\Rightarrow\qquad d\widetilde{\tau} = \Gamma\sqrt{g_{00}}\,d\tau.$$

Thus, we can see from (4.2.2.5) that time dilation will occur under circumstances of either motion, or presence in a potential field, or due to both. Comparison to an equivalent metric in a frame in flat space without motion, we have

$$\delta\widetilde{t} = \delta t\sqrt{1 + 2\frac{L}{L_0}}.$$

If we consider free particle motion, we can see that:

$$U(\boldsymbol{x}) = 0 \qquad\Rightarrow\qquad \delta\widetilde{t} = \delta t\sqrt{1 - \left(\frac{|\boldsymbol{v}|}{c}\right)^2}. \tag{4.3.2.1}$$

On the other hand, for stationary observation, time dilation is caused by potential fields:

$$\boldsymbol{v} = 0 \qquad\Rightarrow\qquad \delta\widetilde{t} = \delta t\sqrt{1 + \frac{2U(\boldsymbol{x})}{mc^2}}. \tag{4.3.2.2}$$

This is confirmed by the theories of gravitational redshift that occurs as monochromatic light of a certain frequency in free space enters a gravitational field. If the time period of the light frequency in presence of a gravitational field is given by T, then according to (4.3.2.2)

$$U = -\frac{GMm}{r} \qquad\qquad U_{r=\infty} = 0,$$

$$T \equiv \delta t \qquad \Rightarrow \qquad T_0 = T_{(U=0)} \equiv \delta\tilde{t},$$

$$T_0 = T\sqrt{1 - \frac{2GM}{c^2 r}} \qquad \Rightarrow \qquad T = \frac{T_0}{\sqrt{1 - \frac{2GM}{c^2 r}}}. \tag{4.3.2.3}$$

If we define the Event Horizon radius as $r_0 = \frac{GM}{c^2}$, then the new frequency in the presence of a gravitational field according to (4.3.2.3) is given by:

$$\nu = \frac{1}{T} = \frac{1}{T_0}\sqrt{1 - 2\frac{r_0}{r}} \qquad \nu_\infty = \frac{1}{T_0},$$

$$\nu = \nu_\infty \sqrt{1 - 2\frac{r_0}{r}} \qquad \Rightarrow \qquad \Delta\nu = \nu_\infty \left(\sqrt{1 - 2\frac{r_0}{r}} - 1 \right). \tag{4.3.2.4}$$

Under a weak potential limit $2U << mc^2$ or $r_0 << r$, we will have:

$$\nu \approx \nu_\infty \left(1 + \frac{U(\boldsymbol{x})}{mc^2} \right) = \nu_\infty \left(1 - \frac{r_0}{r} \right) \qquad \Rightarrow \qquad \Delta\nu \approx \frac{U(\boldsymbol{x})}{mc^2}\nu_\infty = -\frac{r_0}{r}\nu_\infty.$$

Furthermore, we will also have:

$$\frac{\nu_{r1}}{\nu_{r2}} = \sqrt{\frac{1 - 2\frac{r_0}{r_1}}{1 - 2\frac{r_0}{r_2}}}. \tag{4.3.2.5}$$

which is confirmed in any literature on the topic of general relativity [181]. Next, we shall briefly summarize the formulation of relativistic Hamiltonian mechanics on curved spaces.

4.4 Approximations and Limits

When dealing with a relativistically described system, we often are required to apply approximations and limits to conform with the established non-relativistic formulation or the conventional relativistic formulation which uses γ instead of Γ. This will confirm if we are on the right track in our analysis. Special relativity was formulated entirely for free particles. Now we shall explore various ways of applying a weak potential.

4.4.1 Weak potential limit

Under the weak potential approximation, we shall consider the case where

$$U(\boldsymbol{x}) << mc^2. \tag{4.4.1.1}$$

Directly from (4.2.2.10), we can write that

$$\frac{d^2\boldsymbol{x}}{d\tilde{t}^2} = -\frac{1}{m}\left(\frac{dt}{d\tilde{t}}\right)^2 \boldsymbol{\nabla} U(\boldsymbol{x}) \equiv -\frac{c^2}{2}\left(\frac{dt}{d\tilde{t}}\right)^2 \boldsymbol{\nabla} g_{00}(\boldsymbol{x}).$$

Applying weak potential approximation (4.4.1.1), we can say from (4.2.2.5) that

$$\frac{d\tilde{t}}{dt} = \sqrt{1 + 2\frac{L}{L_0}} \approx \sqrt{1 - \left(\frac{v}{c}\right)^2} = \gamma^{-1},$$

$$\therefore \qquad \frac{d^2\boldsymbol{x}}{d\widetilde{t}^2} = -\frac{c^2}{2}\boldsymbol{\nabla}\gamma^2 g_{00}(\boldsymbol{x}).$$

and since we are considering Lorentz transformations of space-time event intervals, and $g_{00} = g_{00}(\boldsymbol{x})$, we can say that

$$dt \longrightarrow d\widetilde{t} = \frac{d\widetilde{t}}{dt}dt \qquad \Rightarrow \qquad g_{00} \longrightarrow \widetilde{g}_{00} = \left(\frac{d\widetilde{t}}{dt}\right)^{-2} g_{00},$$

$$\therefore \qquad \widetilde{g_{00}}(\boldsymbol{x}) = \gamma^2 g_{00}(\boldsymbol{x}).$$

which lets us write the Lorentz-covariant equation of motion:

$$\frac{d^2\boldsymbol{x}}{d\widetilde{t}^2} = -\frac{c^2}{2}\boldsymbol{\nabla}\widetilde{g_{00}}(\boldsymbol{x}) \equiv -\frac{1}{m}\boldsymbol{\nabla}\widetilde{U}(\boldsymbol{x}). \tag{4.4.1.2}$$

Now, we shall look at an alternate formulation with weak potentials.

4.4.2 Semi-relativistic formulation

Another way to describe the non-relativistic approximation $U(\boldsymbol{x}) << mc^2, |\boldsymbol{v}| << c$, of the metric (4.2.2.4) using the expression of γ from (4.2.2.5) is:

$$\mathcal{L} = -mc^2\sqrt{\left(1 - \frac{|\boldsymbol{v}|^2}{c^2}\right) + \frac{2U}{mc^2}} = -mc^2\sqrt{\gamma^{-2} + \frac{2U}{mc^2}} = -mc^2\gamma^{-1}\sqrt{1 + \frac{2U}{mc^2}\gamma^2}.$$

Binomially expanding the expression within the square-root gives us:

$$\mathcal{L} \approx -mc^2\gamma^{-1}\left(1 + \frac{U}{mc^2}\gamma^2\right) = -mc^2\gamma^{-1} - U\gamma,$$

$$\therefore \qquad \mathcal{L} \approx -mc^2\gamma^{-1} - U\gamma. \tag{4.4.2.1}$$

which is different from the form of the Lagrangian employed by Goldstein [173]. However, we must keep in mind that upon applying it into Euler-Lagrange equations, we get the semi-relativistic equation of motion:

$$m\frac{d}{dt}(\gamma\boldsymbol{v}) = -\gamma\boldsymbol{\nabla}U \qquad \xrightarrow{\textit{multiply } \gamma} \qquad m\frac{d^2\boldsymbol{x}}{d\widetilde{t}^2} = -\gamma^2\boldsymbol{\nabla}U. \tag{4.4.2.2}$$

which is the same equation of motion given by (4.4.1.2). Another approximation we can employ here is based on the comparison of the magnitude of factors paired with γ^{-1} and γ in (4.4.2.1) upon binomial expansion. In simple words, if γ^{-1} and γ in (4.4.2.1) are expanded to 1st order:

$$\gamma^{-1} \approx \left(1 - \beta^2\right)^{\frac{1}{2}} \approx 1 - \frac{\beta^2}{2}, \qquad \gamma \approx \left(1 - \beta^2\right)^{-\frac{1}{2}} \approx 1 + \frac{\beta^2}{2}, \qquad \text{where } \beta = \frac{|\boldsymbol{v}|}{c}.$$

we can see that higher order terms from expansion will become significant contributors depending on the factor multiplied to it. Now we can see that for weak potentials

$$mc^2 >> U(\boldsymbol{x}).$$

This means that at least 1st order contribution from γ^{-1} will be significant in $mc^2\gamma^{-1}$. On the other hand, the 1st order contribution fromγ will be insignificant in $U\gamma$. This analysis is elaborated as shown below:

$$\therefore \quad \frac{|\boldsymbol{v}|}{c} = \beta << 1 \quad \Rightarrow \quad \begin{cases} mc^2\gamma^{-1} \approx m\left(c^2 - \dfrac{|\boldsymbol{v}|^2}{2}\right) \\ U\gamma \approx U + \dfrac{U}{2}\beta^2 \end{cases} . \tag{4.4.2.3}$$

Clearly, we can see that in (4.4.2.3), the existence of the β^2 term in the potential energy part of (4.4.2.1) allows us to safely omit a part of of the Lagrangian for the limit $\beta << 1$. This means that we can say:

$$\lim_{\beta<<1} \frac{U}{2}\beta^2 = 0 \qquad \Rightarrow \qquad U\gamma \approx U.$$

Thus, we can say that the semi-relativistic Lagrangian for low velocities can be written as:

$$\therefore \qquad \mathcal{L}_{sr} \approx -mc^2\gamma^{-1} - U. \tag{4.4.2.4}$$

Now this matches the form of the semi-relativistic Lagrangian (4.1.3) employed by Goldstein [173]. We can further proceed to say that if the same pattern of approximation is applied to the semi-relativistic equation of motion (4.4.2.2), we shall have:

$$m\frac{d^2\boldsymbol{x}}{d\tilde{t}^2} = -\gamma^2\boldsymbol{\nabla}U \approx -\left(1+\beta^2\right)\boldsymbol{\nabla}U.$$

Using the same approximation rule (4.4.2.3), we can ignore the term with β^2 to write:

$$\lim_{\beta<<1} \beta^2\boldsymbol{\nabla}U = 0 \qquad \Rightarrow \qquad \gamma^2\boldsymbol{\nabla}U \approx \boldsymbol{\nabla}U,$$

$$\therefore \qquad \lim_{\beta<<1} m\frac{d^2\boldsymbol{x}}{d\tilde{t}^2} = -\boldsymbol{\nabla}U. \tag{4.4.2.5}$$

This equation is thus nearly the same as the usual equation of motion known classically, except for the usage of proper time $\tilde{t}$ in the particle frame instead of t. From this equation, we can write a conserved quantity given as:

$$H = \frac{m}{2}\left|\frac{d\boldsymbol{x}}{d\tilde{t}}\right|^2 + U. \tag{4.4.2.6}$$

As stated, this formulation shall only apply in the low velocity limit for weak potentials. Now we shall look at the relativistic Hamiltonian formulation under weak potentials.

4.4.3 Hamiltonian formulation under weak potential

Under circumstances of a weak potential ($2U(\boldsymbol{x}) << mc^2$) and low momentum $\left(\dfrac{|\boldsymbol{p}|}{mc} \approx 0\right)$, another way to write the relativistic energy (4.2.3.1) is:

$$\mathcal{E} = \sqrt{1 + \frac{2U(\boldsymbol{x})}{mc^2}}\sqrt{|p|^2c^2 + m^2c^4} \approx \left(1 + \frac{U(\boldsymbol{x})}{mc^2}\right)\sqrt{|p|^2c^2 + m^2c^4},$$

$$\Rightarrow \qquad \mathcal{E} \approx \sqrt{|p|^2c^2 + m^2c^4} + U(\boldsymbol{x})\sqrt{\left(\frac{|p|}{mc}\right)^2 + 1} \approx \sqrt{|p|^2c^2 + m^2c^4} + U(\boldsymbol{x}),$$

$$\therefore \qquad \mathcal{H} = \mathcal{E} \approx \sqrt{|p|^2c^2 + m^2c^4} + U(\boldsymbol{x}). \tag{4.4.3.1}$$

Thus, the Hamilton's equations of motion (4.2.3.2) evolve into the form presented in [172]:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} = \frac{c\boldsymbol{p}}{\sqrt{|p|^2 + m^2c^2}}, \\ \dot{\boldsymbol{p}} &= -\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}} = -\boldsymbol{\nabla} U(\boldsymbol{x}). \end{aligned} \tag{4.4.3.2}$$

Now we shall proceed to study the relativistic oscillator and its duality with the Kepler system.

4.5 Bohlin-Arnold Duality

Now that we have properly described the relativistic formulation for classical mechanical systems in a general scalar potential field, we shall now apply this formulation to two important mechanical systems frequently discussed in classical mechanics. They are the Hooke oscillator and Kepler systems. These systems are also dual to each other via a conformal transformation known as the Bohlin-Arnold-Vasiliev transformation.

4.5.1 Relativistic 2D Isotropic Oscillator and Kepler systems

One may ask how such a duality is a matter of concern here in the analysis of Newtonian gravity. While the Kepler potential is known to describe Newtonian gravity, it doesn't seem possible to find a Hooke's law potential that can be described as a result of curved space. The answer lies in the way Hooke's oscillator mechanics are applied in physics; around equilibrium points. In the study of planetary motion, one encounters equilibrium points known as Lagrange points that allow planets to maintain stable orbits. It is locally around these points that one will find single particles to exhibit Hooke oscillatory motion, whose potential function can be described as the curvature of the local space-time.

In a manner similar to (4.2.2.1), the metric of the relativistic gravitational oscillator can be given by:

$$ds^2 = \left(1 + \frac{kr^2}{mc^2}\right) c^2dt^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta \; d\varphi^2\right). \tag{4.5.1.1}$$

The Lagrangian corresponding to (4.5.1.1) according to (4.2.2.4) would be

$$\mathcal{L} = -mc^2\sqrt{1 - \frac{2}{mc^2}\left[\frac{m\left[\dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta \; \dot{\varphi}^2\right)\right]}{2} - \frac{kr^2}{2}\right]}. \tag{4.5.1.2}$$

For planar motion $\theta = \frac{\pi}{2}$, the momenta are given by:

$$\begin{aligned} &p_r = mc\Gamma\dot{r} \qquad p_\varphi = mc\Gamma r^2\dot{\varphi} = l, \\ &\mathcal{E} = p_r\dot{r} + p_\varphi\dot{\varphi} - \mathcal{L} = mc^2\left(1 + \frac{kr^2}{mc^2}\right)\Gamma, \end{aligned} \qquad \text{where} \qquad \Gamma = -\frac{mc^2}{\mathcal{L}}. \tag{4.5.1.3}$$

So the relativistic radial equation would be given by:

$$\frac{d}{dt}\left(\Gamma \dot{r}\right) = -\left(\frac{kr}{m} - r\dot{\varphi}^2\right)\Gamma. \tag{4.5.1.4}$$

while the angular equation is given by:

$$\frac{dp_\varphi}{d\tau} = \frac{d}{d\tau}\left(r^2\dot{\varphi}\Gamma\right) = 0.$$

If we choose the proper time $\widetilde{t}$ as parameter

$$\Gamma = \frac{d\widetilde{t}}{dt} = \sqrt{1 - \frac{2}{mc^2}\left[\frac{m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right)}{2} - \frac{kr^2}{2}\right]}.$$

then we can modify equation (4.5.1.4) in accordance with (4.2.2.10) to:

$$m\left[\frac{d^2r}{d\widetilde{t}^2} - r\left(\frac{d\varphi}{d\widetilde{t}}\right)^2\right] = -kr\left(\frac{dt}{d\widetilde{t}}\right)^2. \tag{4.5.1.5}$$

For small oscillations, according to (4.4.2.1), the relativistic Lagrangian is:

$$\mathcal{L} = -mc^2\gamma^{-1} - \frac{kr^2}{2}\gamma. \tag{4.5.1.6}$$

The Euler-Lagrange equation of motion that we can derive from (4.5.1.6) are:

$$\frac{d}{dt}\left(\gamma \boldsymbol{v}\right) = -\omega^2\gamma\boldsymbol{x} \qquad\qquad \omega^2 = \frac{k}{m}. \tag{4.5.1.7}$$

which does not match the form presented in [171]. However, for small oscillations during which the maximum velocities achieved are relatively small compared to the speed of light, we shall have according to (4.4.2.4) and (4.4.2.5):

$$\mathcal{L}_{sr} \approx -mc^2\gamma^{-1} - \frac{\omega^2}{2}|\boldsymbol{x}|^2, \tag{4.5.1.8}$$

$$\frac{d\widetilde{\boldsymbol{v}}}{d\widetilde{t}} = -\omega^2\boldsymbol{x}, \qquad\qquad \text{where } \omega^2 = \frac{k}{m}. \tag{4.5.1.9}$$

which matches the form presented in [171].

Again, according to (4.2.2.1), the relativistic Kepler system can be given by:

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{4.5.1.10}$$

The Lagrangian corresponding to (4.5.1.10) according to (4.2.2.4) would be

$$\mathcal{L} = -mc^2\sqrt{1 - \frac{2}{mc^2}\left[\frac{m\left[\dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right)\right]}{2} + \frac{GMm}{r}\right]}. \tag{4.5.1.11}$$

For planar motion $\theta = \frac{\pi}{2}$, the momenta are given by:

$$\begin{aligned} &p_r = mc\Gamma \dot{r} \qquad\qquad p_\varphi = mc\Gamma r^2 \dot{\varphi} = l, \\ &\mathcal{E} = p_r \dot{r} + p_\varphi \dot{\varphi} - \mathcal{L} = mc^2 \left(1 - \frac{2GM}{rc^2}\right)\Gamma, \end{aligned} \qquad \text{where} \qquad \Gamma = -\frac{mc^2}{\mathcal{L}}. \tag{4.5.1.12}$$

So the relativistic radial equation would be given by:

$$\frac{d}{dt}\left(\Gamma \dot{r}\right) = -\left(\frac{GM}{r^2} - r\dot{\varphi}^2\right)\Gamma. \tag{4.5.1.13}$$

while the angular equation is given by:

$$\frac{dp_\varphi}{d\tau} = \frac{d}{d\tau}\left(r^2 \dot{\varphi}\Gamma\right) = 0 \qquad \Rightarrow \qquad p_\varphi = r^2 \dot{\varphi}\Gamma = const.$$

showing that the angular momentum is conserved for radial forces. If we choose the proper time $\widetilde{t}$ as parameter

$$\Gamma = \frac{d\widetilde{t}}{dt} = \sqrt{1 - \frac{2}{mc^2}\left[\frac{m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right)}{2} + \frac{GMm}{r}\right]}.$$

then we can modify equation (4.5.1.13) in accordance with (4.2.2.10) to:

$$m\left[\frac{d^2 r}{d\widetilde{t}^2} - r\left(\frac{d\varphi}{d\widetilde{t}}\right)^2\right] = -\frac{GM}{r^2}\left(\frac{dt}{d\widetilde{t}}\right)^2. \tag{4.5.1.14}$$

For small oscillations, according to (4.4.2.1), the relativistic Lagrangian is:

$$\mathcal{L} = -mc^2\gamma^{-1} + \frac{GMm}{r}\gamma. \tag{4.5.1.15}$$

The Euler-Lagrange equation of motion that we can derive from (4.5.1.15) are:

$$\frac{d}{dt}\left(\gamma \boldsymbol{v}\right) = -\frac{GMm}{r^3}\gamma \boldsymbol{x}. \tag{4.5.1.16}$$

Again, for relatively low velocities, we shall have according to (4.4.2.5):

$$\mathcal{L}_{sr} \approx -mc^2\gamma^{-1} + \frac{GMm}{r}, \tag{4.5.1.17}$$

$$\frac{d\widetilde{\boldsymbol{v}}}{d\widetilde{t}} = -\frac{GMm}{r^3}\boldsymbol{x}. \tag{4.5.1.18}$$

We shall now briefly turn our attention to the Bohlin-Arnold duality to demonstrate semi-relativistic Kepler-Hooke duality.

4.5.2 Semi-Relativistic Kepler-Hooke duality

The Kepler-Hooke duality established by Bohlin, Arnold and Vasiliev [36, 30] is a connection between the two mechanical systems which according to Bertrand's Theorem are the only two that are possible with closed, periodic orbits. This duality is established for the classical cases, but are not possible for the relativistic versions of the mechanical systems due to the Γ factors involved in the equations of motion. Here, we show that it is again possible for the semi-relativistic equations of motion given by (4.5.1.9) and (4.5.1.18).

The Bohlin transformation is a conformal map given by:

$$f : z \longrightarrow \xi = (z)^2 = Re^{i\phi} \qquad \Rightarrow \qquad z = \xi^{\frac{1}{2}}. \tag{4.5.2.1}$$

Now we must note that another Noether invariant, the angular momentum will change form under this transformation. We re-parameterize to preserve the form of angular momentum.

$$l = r^2\dot{\theta} = |z|^2\dot{\theta} = |\xi|^2\phi' \qquad \Rightarrow \qquad |\xi|\frac{d\widetilde{\tau}}{d\widetilde{t}}\theta' = |\xi|^2\theta',$$

$$\therefore \quad \widetilde{t} \longrightarrow \widetilde{\tau} : \frac{d\widetilde{\tau}}{d\widetilde{t}} = |\xi|. \tag{4.5.2.2}$$

The velocity and acceleration transformation can be given using (4.5.2.1) and (4.5.2.2):

$$\dot{z} = \frac{1}{2}\frac{|\xi|}{(\xi)^{\frac{1}{2}}}\xi' = \frac{1}{2}\left(\bar{\xi}\right)^{\frac{1}{2}}\xi',$$

$$\ddot{z} = \frac{1}{2}|\xi|\frac{d}{d\widetilde{\tau}}\left\{\left(\bar{\xi}\right)^{\frac{1}{2}}\xi'\right\} = \frac{1}{2}\frac{|\xi|^2}{(\xi)^{\frac{1}{2}}}\xi'' + \frac{1}{4}\left(\xi\right)^{\frac{1}{2}}|\xi'|^2.$$

Thus, the semi-relativistic equation of motion for oscillators (4.5.1.9) eventually becomes:

$$m\left\{\frac{1}{2}\frac{|\xi|^2}{(\xi)^{\frac{1}{2}}}\xi'' + \frac{1}{4}\left(\xi\right)^{\frac{1}{2}}|\xi'|^2\right\} = -k\left(\xi\right)^{\frac{1}{2}},$$

$$\Rightarrow \qquad \xi'' = -\left(\frac{1}{2}|\xi'|^2 + \frac{2k}{m}\right)\frac{\xi}{|\xi|^2}. \tag{4.5.2.3}$$

Using the conserved quantity H from (4.4.2.6) for the oscillator system

$$H = \frac{m}{2}|\dot{z}|^2 + \frac{k}{2}|z|^2 = \frac{m}{4}\left(\frac{|\xi'|^2}{2} + \frac{2k}{m}\right)|\xi|.$$

we can complete the transformation (4.5.2.3) using $\left(\frac{|\xi'|^2}{2} + \frac{2k}{m}\right) = \frac{4H}{m}\frac{1}{|\xi|} = \kappa\frac{1}{|\xi|}$:

$$\therefore \qquad \xi'' = -\left(\frac{|\xi'|^2}{2} + \frac{2k}{m}\right)\frac{\xi}{|\xi|^2} \equiv -\kappa\frac{\xi}{|\xi|^3}. \tag{4.5.2.4}$$

showing that the transformation restores the central force nature of the system and produces the complex version of the Kepler equation (4.5.1.18).

4.6 Relativistic Lienard-type oscillator

In the study of dynamical systems, the Lienard system is a 2nd order differential equation named after French physicist Alfred-Marie Liénard [182]. It is a very generalized way to describe 1-dimensional motion under the influence of scalar potential and damping effects. Such differential equations were used to model oscillating circuits for applications in radios and vacuum tubes. For oscillatory systems, Liénard's Theorem, under certain assumptions assures uniqueness and existence of a limit cycle for the system.

The equation of a damped 1-dimensional relativistic harmonic oscillator is:

$$\gamma^3\ddot{x} + \alpha\gamma\dot{x} + \omega^2 x = 0 \qquad\qquad \gamma = \left(1 - \frac{\dot{x}^2}{c^2}\right)^{-\frac{1}{2}}.$$

In contrast, the damped relativistic Lienard equation is:

$$\gamma^3\ddot{x} + \gamma f(x)\dot{x} + g(x) = 0 \qquad\qquad \gamma = \left(1 - \frac{\dot{x}^2}{c^2}\right)^{-\frac{1}{2}}. \tag{4.6.1}$$

Under reparametrization $t \longrightarrow d\widetilde{t} = \gamma^{-1}dt$, we have $x' = \frac{dx}{d\widetilde{t}} = \gamma\frac{dx}{dt} = \gamma\dot{x}$, letting us write:

$$\frac{d}{dt}(\gamma\dot{x}) = \gamma\ddot{x} + \gamma^3\frac{\dot{x}^2}{c^2}\ddot{x} = \gamma^3\ddot{x}\left(1 - \frac{\dot{x}^2}{c^2} + \frac{\dot{x}^2}{c^2}\right) = \gamma^3\ddot{x},$$

$$\therefore \qquad \gamma^3\ddot{x} = \frac{d}{dt}(\gamma\dot{x}) = \frac{d}{dt}x' = \gamma^{-1}\frac{d}{d\widetilde{t}}x' = \gamma^{-1}x''.$$

Thus, under reparametrization, (4.6.1) becomes

$$x'' + \gamma\left[f(x)x' + g(x)\right] = 0. \tag{4.6.2}$$

Remember that

$$\gamma^{-2} = 1 - \left(\frac{\dot{x}}{c}\right)^2 = 1 - \gamma^{-2}\left(\frac{x'}{c}\right)^2 \qquad \Rightarrow \qquad \gamma = \sqrt{1 + \left(\frac{x'}{c}\right)^2}.$$

Thus, when $\dfrac{x'}{c} \longrightarrow 0$, we will have $\gamma \longrightarrow 1$, which is the same as when $\dfrac{\dot{x}}{c} \longrightarrow 0$.

4.6.1 Integrability and Relativistic Chiellini condition

Now, the relativistic Chiellini condition is given by:

$$\frac{d}{dx}\left(\frac{g}{f}\right) = -\alpha(1+\alpha)\gamma f(x). \tag{4.6.1.1}$$

Thus, using (4.6.1.1), and using the integrating factor $\Omega = \int^{\widetilde{t}} d\tau \ \gamma f(x)$, we can rewrite (4.6.2) as:

$$x'' + \gamma f(x)x' - \frac{1}{\alpha(1+\alpha)}\left(\frac{g}{f}\right)\frac{d}{dx}\left(\frac{g}{f}\right) = 0. \tag{4.6.1.2}$$

$$\Rightarrow \quad 2e^{\Omega}\left[x'x'' + \gamma f(x)(x')^2\right] - \frac{2e^{\Omega}}{\alpha(1+\alpha)}\left(\frac{g}{f}\right)\left\{\frac{d}{dx}\left(\frac{g}{f}\right)x'\right\} = 0,$$

$$\Rightarrow \quad \frac{d}{d\tilde{t}}\left[\left(\mathrm{e}^{\Omega}(x')^2\right) - \frac{\mathrm{e}^{\Omega}}{\alpha\,(1+\alpha)}\frac{g}{f}x' - \frac{\mathrm{e}^{\Omega}}{\alpha\,(1+\alpha)}\left(\frac{g}{f}\right)^2\right] \\ + \frac{1}{\alpha\,(1+\alpha)}\left(\frac{g}{f}\right)\left[\frac{d}{d\tilde{t}}\left(\mathrm{e}^{\Omega}x'\right) + \left(\frac{g}{f}\right)\frac{d\mathrm{e}^{\Omega}}{d\tilde{t}}\right] = 0.$$

Now, using the Chiellini condition (4.6.1.1), and referring to (4.6.2) we can see that

$$\begin{aligned} \frac{d}{d\tilde{t}}\left(\mathrm{e}^{\Omega}x'\right) + \left(\frac{g}{f}\right)\frac{d\mathrm{e}^{\Omega}}{d\tilde{t}} &= \mathrm{e}^{\Omega}\left[x'' + \left\{x' + \left(\frac{g}{f}\right)\right\}(\gamma f(x))\right], \\ &= \mathrm{e}^{\Omega}\left[x'' + \gamma\left(f(x)x' + g(x)\right)\right] = 0, \end{aligned}$$

$$\therefore \qquad \frac{d}{d\tilde{t}}\left[\mathrm{e}^{\Omega}\left\{(x')^2 - \frac{1}{\alpha\,(1+\alpha)}\frac{g}{f}\left(x' + \frac{g}{f}\right)\right\}\right] = 0.$$

Thus, we have a conserved quantity given by

$$I = \mathrm{e}^{\Omega}\left[\gamma^2(\dot{x})^2 - \frac{1}{\alpha\,(1+\alpha)}\frac{g}{f}\left(\gamma\dot{x} + \frac{g}{f}\right)\right], \qquad \Omega = \int^{\tilde{t}} d\tau\ \gamma f(x). \tag{4.6.1.3}$$

Thus, we have a conserved quantity for a relativistic Liénard system. To solve it for the damped oscillator, we write $f(x) = \kappa$, $g(x) = x$.

4.6.2 Metric and Lagrangian

To deduce the metric from the relativistic equation of motion for a damped system (4.6.1), we shall execute a more elaborate 5-step procedure than the 3-step procedure for undamped systems, given by:

1. Convert the equation (4.6.1) to the non-relativistic version (with $\gamma = 1$).

$$\ddot{\boldsymbol{x}} + \alpha\dot{\boldsymbol{x}} + \frac{1}{m}\boldsymbol{\nabla}U = 0. \tag{4.6.2.1}$$

2. Determine the reparametrization factor e^{Ω} by converting (4.6.2.1) to the form (4.2.2.12).

$$\frac{d}{dt}\left(\mathrm{e}^{\Omega}\dot{\boldsymbol{x}}\right) + \frac{\mathrm{e}^{\Omega}}{m}\nabla U = 0, \qquad \Omega = \int^{t} dt'\ \alpha,$$

$$\Rightarrow \qquad \boldsymbol{x}'' = -\frac{\mathrm{e}^{2\Omega}}{m}\boldsymbol{\nabla}U = -\frac{c^2\mathrm{e}^{2\Omega}}{2}\boldsymbol{\nabla}g_{00}, \qquad \boldsymbol{x}' = \mathrm{e}^{\Omega}\dot{\boldsymbol{x}}, \qquad d\tau = \mathrm{e}^{-\Omega}dt. \tag{4.6.2.2}$$

3. Deduce the undamped potential U from (4.6.2.2) by factoring out $\mathrm{e}^{2\Omega}$.

4. Comparing (4.6.2.2) to (4.2.2.13), we can formulate the damped effective Lagrangian L_d for reparameterized time τ, just as (4.2.2.8) can be derived from (4.2.2.13).

$$\boldsymbol{x}'' = -\frac{c^2\mathrm{e}^{2\Omega}}{2}\boldsymbol{\nabla}g_{00} \quad \longrightarrow \quad L_d = \frac{m}{2}\left[\left|\boldsymbol{x}'\right|^2 - \mathrm{e}^{2\Omega}c^2 g_{00}(\boldsymbol{x})\right],$$

$$L_d = \frac{m}{2}\mathrm{e}^{2\Omega}\left[\left|\dot{\boldsymbol{x}}\right|^2 - c^2 g_{00}(\boldsymbol{x})\right]. \tag{4.6.2.3}$$

5. Deduce the classical damped effective Lagrangian L_{ed} from (4.6.2.3) by multiplying $\mathrm{e}^{-\Omega}$, and write the damped metric ds_d^2 from it.

$$\begin{aligned} \text{relativistic :} \qquad & dS = d\tau\ \mathcal{L} = \mathrm{e}^{-\Omega} dt\ \mathcal{L} \\ \text{effective classical :} \qquad & dS_{eff} = d\tau\ L_d = \mathrm{e}^{-\Omega} dt\ L_d = dt\ L_{ed} \end{aligned}$$

$$L_d = -\frac{m}{2}\left(\frac{ds}{d\tau}\right)^2 = -\frac{m}{2}\mathrm{e}^{2\Omega}\left(\frac{ds}{dt}\right)^2,$$

$$L_{ed} = \mathrm{e}^{-\Omega} L_d = -\frac{m}{2}\mathrm{e}^{-\Omega}\left(\frac{ds}{d\tau}\right)^2 = -\frac{m}{2}\left(\frac{ds_d}{dt}\right)^2 = \frac{m}{2}\mathrm{e}^{\Omega}\left[\left|\dot{\boldsymbol{x}}\right|^2 - c^2 g_{00}(\boldsymbol{x})\right],$$

$$ds_d^2 = \mathrm{e}^{\Omega}\left[c^2 g_{00}(\boldsymbol{x}) dt^2 - |d\boldsymbol{x}|^2\right]. \tag{4.6.2.4}$$

Looking back at (4.6.1), we can take the non-relativistic version of the equation and say that

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

$$\therefore \qquad \frac{d}{dt}\left(\mathrm{e}^{\Omega}\dot{x}\right) = -\mathrm{e}^{\Omega} g(x) \qquad\qquad \Omega = \int^t d\tilde{t}\ f(x).$$

If we reparameterize as $d\tau = \mathrm{e}^{-\Omega} dt$ then we will have using the non-relativistic version of Chiellini condition (4.6.1.1)

$$\frac{d}{dx}\left(\frac{g}{f}\right) = -\alpha\left(1+\alpha\right) f(x),$$

$$\frac{d^2x}{d\tau^2} = -\mathrm{e}^{2\Omega} f(x)\frac{g}{f} = \frac{\mathrm{e}^{2\Omega}}{2\alpha\left(1+\alpha\right)}\frac{d}{dx}\left[\left(\frac{g}{f}\right)^2\right] = -\frac{c^2}{2}\frac{d}{dx}\left[\mathrm{e}^{2\Omega}\left\{1 - \frac{1}{c^2\alpha\left(1+\alpha\right)}\left(\frac{g}{f}\right)^2\right\}\right],$$

$$\therefore \qquad \frac{d^2x}{d\tau^2} = -\frac{c^2}{2}\frac{d}{dx}\left[\mathrm{e}^{2\Omega}\left\{1 - \frac{1}{c^2\alpha\left(1+\alpha\right)}\left(\frac{g}{f}\right)^2\right\}\right]. \tag{4.6.2.5}$$

Arguing that $x = x(t)$ and from (4.6.2.5) we have the undamped potential $U(\boldsymbol{x})$ and $g_{00}(\boldsymbol{x})$:

$$U(\boldsymbol{x}) = -\frac{m}{2\alpha\left(1+\alpha\right)}\left(\frac{g}{f}\right)^2 \qquad \Rightarrow \qquad g_{00}(\boldsymbol{x}) = \left[1 - \frac{1}{c^2\alpha\left(1+\alpha\right)}\left(\frac{g}{f}\right)^2\right]. \tag{4.6.2.6}$$

Using (4.6.2.6), the damped effective Lagrangian according to (4.6.2.3), is given from (4.6.2.5) as:

$$L_d = \frac{m\mathrm{e}^{2\Omega}}{2}\left[\left(\frac{dx}{dt}\right)^2 - c^2\left\{1 - \frac{1}{c^2\alpha\left(1+\alpha\right)}\left(\frac{g}{f}\right)^2\right\}\right].$$

Now, we can rewrite this as

$$\Rightarrow \qquad L_{ed} = \mathrm{e}^{-\Omega} L_d = \frac{m}{2}\mathrm{e}^{\Omega}\left[\left(\frac{dx}{dt}\right)^2 - c^2\left\{1 - \frac{1}{c^2\alpha\left(1+\alpha\right)}\left(\frac{g}{f}\right)^2\right\}\right]. \tag{4.6.2.7}$$

and finally, the metric with drag factor from (4.6.2.7) is (4.6.2.4) given by:

$$\left(\frac{ds}{dt}\right)^2 = -\frac{2}{m}L_{ed} \qquad \Rightarrow \qquad ds^2 = \mathrm{e}^{\Omega}\left\{1 - \frac{1}{c^2\alpha\left(1+\alpha\right)}\left(\frac{g}{f}\right)^2\right\} c^2 dt^2 - \mathrm{e}^{\Omega} dx^2. \tag{4.6.2.8}$$

where we can restore the integrating factor Ω back to the relativistic version $\Omega = \int^t d\widetilde{t}\, \gamma f(x)$.

This further goes to show that damped mechanical systems can be described by a metric that is spatially isotropic with a non-unity co-efficient. Furthermore, if it is a solution of Einstein's equations, then we should have:

$$\mathrm{e}^{-2\Omega} = 1 - \frac{1}{c^2\alpha\,(1+\alpha)}\left(\frac{g}{f}\right)^2.$$

so from (4.6.2.8) we get the metric:

$$ds^2 = \frac{c^2dt^2}{\mathrm{e}^{\Omega}} - \mathrm{e}^{\Omega}dx^2.$$

This topic will be elaborated upon in greater detail in another project where such spaces are studied as damped mechanical systems. [1]

[1] Contact Hamiltonian mechanics [183] extend symplectic Hamiltonian mechanics [184], geometrically describing non-dissipative and dissipative systems, eg.: thermodynamics [185], mesoscopic dissipative mechanical systems [186], and mechanical systems drawing energy from a reservoir.

The Hamiltonian is effectively provided by I in (4.6.1.3). If we define a new variable s, as in [187], ignoring the decay-countering factor e^{Ω} gives the decaying Hamiltonian embedded in (4.6.1.3):

$$s := -\frac{1}{\alpha\,(\alpha+1)}\left(\frac{g}{f}\right)\gamma\dot{x}, \qquad H = (\gamma\dot{x})^2 - \frac{1}{\alpha\,(1+\alpha)}\left(\frac{g}{f}\right)^2 + s.$$

From the relativistic non-decaying Lagrangian of (4.6.2.8), the relativistic momentum under weak potential approximates to $p \approx \gamma\dot{x}$, which we can replace in s and H defined above to rewrite them, from which we can see that

$$\dot{s} = f(x)\left[\frac{\partial H}{\partial p}p - H\right].$$

However, as stated in [187], unless $f(x) = const$, we cannot recast the Liénard equation in contact form, which is simply the damped oscillator.

Chapter 5

Comments and Discussions

5.1 Summary

5.1.1 Chapter 2

The Jacobi metric formulation reduces dynamics from autonomous Lagrangian and Hamiltonian perspective on a $n+1$ dimensional space with potential function to an equivalent free particle geodesic in n dimensions. All aspects of integrability and first integrals are preserved under such reductions. The relativistic Jacobi metric was shown to derive from the space-time curve, preserving angular momentum as a conserved quantity, and acting as a conformally flat metric for cases like Kepler or n-body problems.

Although the relativistic Jacobi metric appears different from the classical version derived from the classical Lagrangian and Hamiltonian, applying non-relativistic approximations to the relativistic version shows that the two forms are equivalent. Thus, we deduced the Jacobi metric in relativistic and non-relativistic form for various metrics: Taub-NUT space, Bertrand space-time and the Kerr space-time. The Hamiltonian and Lagrangian of Jacobi metrics possess a conformal factor and the classical Hamiltonian equates to unity. Such a procedure can cast the TeVeS theory into the form of a Kaluza-Klein construction [188].

So far, the Jacobi-metric could be formulated only for autonomous systems due to a conserved quantity, the Hamiltonian, conjugate to the cyclical co-ordinate, time. However, such convenience is denied for time-dependent systems. In such circumstances, the Eisenhart-Duval lift proves useful, by providing a dummy variable along an extra dimension, and thus, a conserved quantity. This gives us a momentum equation from which we can define a metric for the unit momentum sphere, and thus, the Jacobi metric for time-dependent systems.

In the study of the Kepler problem, such a transformation for a particular energy level, combined with Bohlin's canonical transformation, converts the isotropic oscillator to the Kepler system. Houri's canonical transformation is found to be incomplete without Milnor's momentum inversion map, which preserves the form of geodesic flows as identical to that of the Kepler problem. Alternatively, when the Kepler equation is parameterized with an eccentric anomaly, the resulting dynamics was shown to resemble the motion of a perturbed oscillator.

Quite a few details of Jacobi-Maupertuis formulation have been less studied. For example, the Maupertuis principle can be used in the construction of the theory of many-valued functionals, which arises naturally in the study of the motion of charged particle in a scalar potential field and magnetic field [189]. It would be interesting to extended this project to the study of integrable magnetic geodesic flows [32, 33]. Recently this has been extended in [190] to present a modern outlook to describe the Maupertuis principle's mechanism using classical integrable dynamical systems. This mechanism yields integrable geodesic flows and integrable systems associated with curved spaces. In fact other related topics like the formulation of the Jacobi metric for time-like geodesics and its application to curved space-time [6], applications of geodesic instabilities for the planar gravitational three-body problem [191] should get more attention. It would be fascinating to apply this analysis to the generalized MICZ-Kepler problem.

5.1.2 Chapter 3

Here, we start by applying the bottom-up approach of emergent gravity to (Euclidean) Schwarzschild solution, which we dub as emergent Schwarzschild, describing a Ricci-flat, although not a Kähler manifold, thus, not admitting a natural symplectic structure. The best alternative (as utilized by Etesi and Hausel [192]), was to consider the (anti) self-dual harmonic two-forms on the space and define a Poisson algebra determined by the self-dual harmonic two-form. A magnetic mass (and an electric mass) at the origin seems to violate the Jacobi identity of the underlying Poisson algebra, which can be circumvented by going to Euclidean signature and using Kruskal-Szekeres coordinates. Therefore, the Schwarzschild instanton remained a challenge for the bottom-up approaches of emergent gravity.

A suitable Darboux chart was found for the emergent Schwarzschild solution, for which the Jacobi identity is locally satisfied for the symplectic $U(1)$ gauge fields emergent from the metric as well as the Bianchi identity for the vector fields. We set up the Seiberg Witten map between the commutative and non-commutative (NC) descriptions, executing a thorough geometrical engineering for the instanton solution. The two instantons forming the emergent Schwarzschild solution were found to belong to different gauge groups: $SU(2)_L$ and $SU(2)_R$, so they cannot decay into a vacuum, explaining it's stability against perturbation, which might be generic for any Ricci-flat four manifold as ours.

It is fascinating to investigate a charged black hole solution in this bottom-up approach of emergent gravity. Two kinds of 4D and 2D Extremal Black Holes (EBH) are suggested to exist in nature in [193]; the first kind, with zero entropy, is obtained by taking the extreme limit, followed by the boundary limit starting from general non-extremal configuration, and the second kind, which still holds the topological configuration of Non EBH (NEBH) and satisfying BH entropy formula, is obtained by applying the limits in reverse order. These two kinds of EBHs have different intrinsic thermodynamical properties due to solutions classified by different topological characteristics. The Euler characteristic for the first kind is zero, and for the second, it is 2 or 1 for 4D or 2D EBHs respectively. It will be interesting to see if we can explain such a space-time topology change using a well-defined mechanism inspired by the emergent gravity approach set up by one of the authors in [194].

Next, we have obtained the general Darboux-Halphen system as a reduction of the self-dual Yang-Mills system, which can be transformed to a third-order system, resembling the classical Darboux-Halphen system with a common additive terms. Furthermore, the transformed system can be further reduced to a constrained non-autonomous, non-homogeneous dynamical system. This dynamical system becomes homogeneous for the classical Darboux-Halphen case, studied in the context of self-dual Einstein's equations for Bianchi IX metrics. A Lax pair and Hamiltonian for this reduced system is derived and the solutions for the system are prescribed in terms of hypergeometric functions.

Two different approaches were shown to lead up to the classical Darboux-Halphen system. One starts from the anti-self-dual Bianchi-IX metric, while the other starts with a reduced self-dual Yang-Mills (SDYM) equation, taking only the diagonal elements of the resulting matrix equation. When starting with SDYM gauge fields, it is clear why we cannot always reliably find a metric or its vierbeins that correspond to the generalized DH system. The classical configuration is a typical prototype where it is possible, as was evident when we could obtain it from the self-dual Bianchi-IX metric. We have computed the curvature, confirmed the Ricci-flatness of the self-dual cases, and shown that the classical Darboux-Halphen exhibits Ricci flow for a modified Bianchi-IX system. This system was found to satisfy the Chazy equation as well, and is strongly related to other systems of differential equations, such as the Ramanujan and Ramamani systems.

It is still challenging to find other integrable systems of number theoretic importance. Another useful direction could be to compare solutions of DH type systems (3.3.1.8) and (3.3.2.11) using moving monodromy methods. We are also trying to find interesting 1+1 or 2+1 dimensional DH type systems that are solvable using inverse scattering transform which can be studied to uncover several widely known aspects of integrability. Other samples could be various scalar flat Kähler metrics, namely the LeBrun metric with $U(1)$ isometry that contains Gibbons-Hawking, Real heaven and Burns metric as special limits were used to test the bottom-up approach of emergent gravity [66]. Some important issues for the test of emergent gravity [80] might be clarified by using monodromy evolving deformation [195] on Plebanski type self dual Einstein equations, which are actually the EOM obtained from the 2-dim chiral $U(N)$ model in the large N limit and studying integrability.

Finally, we deal with the Taub-NUT metric, a special case of the anti-self-dual Bianchi-IX spaces [116], whose emergent nature and connection with dynamical systems are discussed in [37]. They are derived by applying the settings for this case to the classical Darboux-Halphen system and solving the resulting dynamical equations. We have shown that the Taub-NUT is comparable to Euclideanized Bertrand space-time with magnetic fields due to shared geometry and conserved quantities, and a dual configuration as either Oscillator or Kepler systems. Identical conserved quantities are related to identical symmetries and Killing tensors embedded within, such as the Killing-Stäckel and Killing-Yano tensors, embedded as co-efficients within the Laplace-Runge-Lenz and angular momentum vectors respectively. The Killing-Yano tensors exhibit quaternionic algebra, hinting at a link between them and hyperkähler structures matching the form of the KY tensors derived from the angular momentum, confirming that they are the same for Taub-NUT. Since space-time symmetries are unaffected by Euclideanization, we can expect that all properties arising from shared symmetries are also exhibited by Bertrand space-times with magnetic fields.

In special situations, self-dual Einstein Bianchi-IX metrics reduce to Taub-NUT de Sitter metric with two parameters of the biaxial solutions respectively identified as the NUT parameter and the cosmological constant. The Taub-NUT is anti-self dual, and expectedly, Ricci-flat with topological invariants to compare with other possible diffeomorphically equivalent Ricci-flat manifolds. According to Kronheimer classifications [196, 197] all 4-dimensional hyperkähler metrics like Taub-NUT are anti-self dual, so the hyperkähler quotient construction, due to Hitchin, Karlhede, Lindstrom and Rocek [198] carries an anti-self dual conformal structure, allowing Penrose's Twistor theory [199] techniques to be applied in this case.

Recent work in emergent gravity [66] aims to construct a Riemannian geometry from $U(1)$ gauge fields on a noncommutative space-time. This construction is invertible to find corresponding $U(1)$ gauge fields on a (generalized) Poisson manifold given a metric (M, g).

Detailed tests [80] of the emergent gravity provide explicit solutions in both gravity and gauge theory perspectives. Symplectic $U(1)$ gauge fields were derived starting from the Eguchi-Hanson metric in 4-dimensional Euclidean gravity, precisely reproducing $U(1)$ gauge fields of the Nekrasov-Schwarz instanton previously derived from the top-down approach. To clarify the role of noncommutativity of space-time in resolving space-time singularities [194] in general relativity, the prescription was inverted. A gravitational metric was derived from the Braden-Nekrasov $U(1)$ instanton defined in ordinary commutative space-time, just to show that the Kähler manifold determined by the Braden-Nekrasov instanton exhibits a space-time singularity, while the Nekrasov-Schwarz instanton gives rise to a regular geometry in the form of Eguchi-Hanson space.

We can speculate the possibility of getting $U(1)$ gauge fields in the same way from the Taub-NUT metric. A critical difference from the Eguchi-Hanson metric [88] is that the Taub-NUT metric (3.5.1.10) is locally asymptotic at infinity to $\mathbb{R}^3 \times \mathbb{S}^1$, belonging to the class of asymptotically locally flat (ALF) spaces. Thus, the Hopf coordinates cannot represent the Taub-NUT metric, and it is difficult to naively generalize the same construction to ALF spaces. According to gauge theory, it may be related to ALF spaces arising from NC monopoles [200] whose underlying equation is defined by an $\mathbb{S}^1$-compactification of the self(anti)-dual-instanton equation, the so-called Nahm equation. A possible inclusion of Taub-NUT in the bottom-up approach of emergent gravity will be discussed in [201]. Only a special choice of the NUT parameter gives us a regular metric, but singularities are generally found at either end of the 4-dim radial coordinate. In the most generic case, for a particular choice of the azimuthal angle period, one can get away with the bolt-singularity. The NUT singularity (co-dimension 4 orbifold singularity) stays, possibly admitting an M theory interpretation associated with the corresponding non-abelian gauge symmetries [202].

Recently, Ricci flat metrics of ultrahyperbolic signature were constructed [203] with l-conformal Galilei symmetry, involving an AdS_2 part reminiscent of the near horizon geometry of extremal black holes. Similarly, it should be interesting to see if Taub-NUT spaces are associable with geodesics describing second order dynamical systems. Perhaps the most interesting issue will be to explore if we can conjecture something like "Taub-NUT/CFT" correspondence.

5.1.3 Chapter 4

The form of relativistic Lagrangian and equations of motion in [171] also used by Goldstein [173], do not match with the formulations derived directly from solving for the geodesic from the space-time metric. The latter formulation clearly reproduces the time dilation effects in a potential field as observed in the phenomena of gravitational redshifts [181], while the former does not. The Hamiltonian formulation shown in [172] applies only for weak potentials and low momentum. We effectively managed to write a Lorentz-covariant form of equations of motion in (4.2.2.16) for geodesics that does not match the classic non-relativistic form of such equations and describe a relativistic deformation of the Euler-Lagrange equation.

Since the familiar Lorentz transformation was designed for the special case of special relativity, where we deal with free particles without any potentials, it will not suffice in a more general case where a particle accelerates under the influence of potentials. The solution was to formulate a modified Lorentz transformation that will locally leave the metric invariant. The fact that such a transformation is limited to work locally is not surprising, given that according to the Equivalence principle, a curved space is locally diffeomorphically equivalent to a flat space, where a regular Lorentz transformation would be valid.

The Bohlin-Arnold duality [36, 30] is invalid for the relativistic oscillator under this direct formulation due to the Γ factors in the equation of motion. However, it is possible to use a semi-relativistic approximation for weak potential and low velocity to produce the type of semi-relativistic Lagrangian in [171, 173]. Using such approximation, and replacing the time with the proper time in the particle frame, the semi-relativistic equation of motion takes the classical form, allowing the Bohlin-Arnold duality to be valid under this approximation.

We have written the relativistic version of the Liénard equation and Chiellini integrability condition, and deduced its conserved quantity, Lagrangian and metric. The conserved quantity was derived after first redefining the relativistically. The metric derived for the Liénard oscillator implies spatially isotropic metrics with non-unity co-efficient function. Thus, if the Schwarzschild, and most solutions to Einstein's equations are rewritten to spatially isotropic co-ordinates, they shall be found to exhibit damped motion, implying that dynamics of such solutions can be examined as some form of the relativistic Liénard mechanical system.

So far, this article helps to revise and generalize our fundamental formulation of relativity, while showing how familiar results can still be reproduced under suitable approximations. The results of this chapter can be applied in future along the direction of canonical ADM gravity and relativistic quantum mechanics as covered in articles like [204, 205] respectively.

5.2 Future Plans: Superintegrable and Dynamical Systems on Curved Spaces

When solving a mechanical system, its overall motion is decomposable into independent motion along each of its degrees of freedom, described by independent ordinary differential equations (ODEs). First integrals or conserved quantities of an autonomous system are functions that maintain a constant value respective to the system. They play a central role in the theory of such ODEs by reducing the effective dimensionality of the problem being dealt with via a change of variables. In effect, each first integral is a solution of the equation along a degree of freedom, having which reduces the number of independent ODEs one has to solve by one, simplifying the problem.

If a sufficient number of first integrals are known, the dynamical system can be completely solved, and is referred to as an integrable system. If the number of available first integrals exceeds the number of degrees of freedom, then the dynamical system is labelled a "superintegrable" system. The availability of additional constants of motion reduces the trajectories of the superintegrable system to lower dimensional submanifolds of the Arnold-Liouville tori. Classical trajectories are closed curves in the case of maximally superintegrable systems, where the number of globally independent integrals of motion increases to $2n - 1$. Bertrand's theorem states that all bounded trajectories are closed only for two central potentials: the Kepler, and the isotropic oscillator, providing a complete classification of 3-D superintegrable systems with central potentials.

A compact and connected n-dimensional submanifold of a phase space determined by n involutive first integrals is topologically equivalent to a n-dimensional torus via the celebrated Arnold-Liouville theorem (in the general non-compact case, it is the product of a torus and Euclidean space). Since there is no unique way to compute the first integrals, we employ various methods to deduce them, which play a vital role in the probing of a dynamical system. In this manner, we have successfully obtained the superintegrability structure of many 3 or 4 dimensional systems.

Generically all constants of motion of integrable and superintegrable systems are the direct consequence of symmetries (in most cases, hidden symmetries); making it convenient to study these symmetries geometrically (ie., symplectic formalism and Lie algebra of vector fields). It also helps to study higher-order constants of motion; ie. Killing tensors K for higher values of p. The Eisenhart lift has been related to the properties of Killing tensors defined on the Riemannian space; making it also a topic worth studying. We are exploring various geometrical techniques, such as master symmetry, action-angle method, Kaluza-Klein reduction, etc, to study superintegrable systems. Furthermore, we wish to use the geometric formalism introduced long ago by Eisenhart to explore the notion of superintegrability. The concept of superintegrability can be further extended to geodesic motion over the Einstein-Sasaki metric and the primary tool to that end are Killing and Killing-Yano tensors.

One of the challenges of Hamiltonian dynamics is to provide adequate mathematical tools to describe chaotic dynamics in a system combining both regular and chaotic components. There exist rigorous results that prove that the underlying systems most of the trajectories are stable for all the time, it is the so well known KAM Theorem (Kolmogorov-Arnold-Moser) or they are stable for very long times (Nekhoroshev). The mathematics of these areas are tackled with a large set of non trivial and important tools as well as a bunch of consolidated methods in dynamical systems: hyperbolic invariant objects and their connections (limit cycles, isochrons, whiskered tori, NHIM, homoclinic orbits, etc.), normal forms, averaging methods, KAM theorem, Lie symmetries etc. These involve both theory and numerics. My targeted area is celestial mechanics, dynamical aspects of general relativity and cosmology.

5.2.1 Finsler geometry

Dynamical and Integrable Systems have so far been extensively studied in a classical and non-relativistic description, where the system is defined by a classical Lagrangian that is quadratic in velocity with scalar potentials on flat space. However, the study of dynamical and integrable systems on curved spaces has not received sufficient attention. In this general relativistic approach, we can describe curved spaces as perturbations created by potentials about flat space, and the relativistic Lagrangian defining the system as the square-root of the classical Lagrangian.

Under low velocity and weak potential approximations, the relativistic Lagrangian and the mechanics that derive from it transforms into the classical equivalents. An attempt to study relativistic dynamical and integrable systems in this approach has been made in [206]. However, some first integrals that have been deduced classically cannot be derived on curved spaces, such as the Laplace-Runge-Lenz vector.

Finsler geometry is an extension of Riemannian metric geometry, based on a general length measure L for curves γ formulated as:

$$L\left[\gamma\right] = \int F(\gamma, \dot{\gamma}).$$

In 1941, G. Randers [19] introduced a Finsler metric by modifying a Riemannian metric $g = g_{ij}\, dx^i \otimes dx^j$ by a linear term $b = b_i(x)dx^i$, the resulting norm on the tangent space is given by

$$F(x, y) = \sqrt{g_{ij}(x)y^i y^j} + b_i(x)y^i, \qquad y = y^i \partial_{x^i} \in T_x M.$$

Now, we have so far seen that when measuring the length of a curve between two points in a space-time, the form of the relativistic Lagrangian as shown in [206] is similar upto the term in square roots of Rander's Finsler metric.

$$l_{12} = \int_1^2 ds = \int_1^2 d\tau \sqrt{g_{ij}(x) y^i y^j}.$$

When dealing with space-times involving vector-potential terms, such as stationary space-times, it is possible under approximations to write the relativistic Lagrangian in Rander's form of the Finsler metric. I intend to study the relativistic mechanics and dynamics of such systems in the Finsler geometry context.

5.2.2 3-body problems on curved spaces in the post-Newtonian approximation

Recently, the detection of gravitational waves was a significant event in the study of general relativity, experimentally confirming the existence of gravitational waves and the accuracy of theories describing them. The detected gravitational waves were produced by the merging of a pair of black holes, which describes a 2-body problem on curved space. I wish to go a step further and proceed to the study of a 3-body problem on curved space, with the potentials acting as perturbations in a post-Newtonian approximation, and the resulting gravitational waves.

We wish to study the n-body problem in spaces of constant curvature, started by Diacu [207, 208], who obtained the equations of motion for an arbitrary n number of bodies in 2008 [209]. This provided new criteria to determine the geometrical nature of physical space. The problem for the case $n = 2$ was independently proposed by Bolyai and Lobachevsky, who founded hyperbolic geometry.

In the usual linearized theory of gravity, the perturbations about the flat space-time metric are taken upto first order approximation. In the post-Newtonian limit, we go further beyond, from the second order onwards. Furthermore, since the study of gravitational waves involves time dependent perturbations, it is necessary to apply the Eisenhart-Duval lift to them to study dynamics on such spaces. I intend to study dynamics and gravitational waves of 3-body problems by combining the post-Newtonian limit with the Eisenhart-Duval lift.

Chapter 6

Appendices

6.1 t'Hooft symbols

Matrices representing the t'Hooft symbols would be given by :

$$\eta^{(+)1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \eta^{(+)2} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \eta^{(+)3} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{6.1.1}$$

$$\eta^{(-)1} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \eta^{(-)2} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \eta^{(-)3} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{6.1.2}$$

which obey the following relations between themselves

$$\sum_{i=1}^{3} \eta^{(\pm)i}_{\mu\nu}\eta^{(\pm)i}_{\lambda\gamma} = \delta_{\mu\lambda}\delta_{\nu\gamma} - \delta_{\mu\gamma}\delta_{\nu\lambda} \pm \varepsilon_{\mu\nu\lambda\gamma} \tag{6.1.3}$$

$$\left[\eta^{(\pm)i}, \eta^{(\pm)j}\right]_{\mu\nu} = -2\epsilon^{ijk}\eta^{(\pm)k}_{\mu\nu} \tag{6.1.4}$$

$$\left[\eta^{(\pm)i}, \eta^{(\mp)j}\right]_{\mu\nu} = 0 \quad \Rightarrow \quad \eta^{(\pm)i}_{\mu\rho}\eta^{(\mp)j}_{\rho\nu} = \eta^{(\pm)j}_{\nu\rho}\eta^{(\mp)i}_{\rho\mu} \tag{6.1.5}$$

$$\left\{\eta^{(\pm)i}, \eta^{(\pm)j}\right\}_{\mu\nu} = -2\delta^{ij}\delta_{\mu\nu} \tag{6.1.6}$$

$$\left\{\eta^{(\pm)i}, \eta^{(\mp)j}\right\}_{\mu\nu} = 0 \quad \Rightarrow \quad \eta^{(\pm)i}_{\mu\nu}\eta^{(\mp)j}_{\mu\nu} = 0 \tag{6.1.7}$$

$$\therefore \quad \eta^{(\pm)i}_{\mu\lambda}\eta^{(\pm)j}_{\nu\lambda} = \delta^{ij}\delta_{\mu\nu} + \epsilon^{ijk}\eta^{(\pm)k}_{\mu\nu} \tag{6.1.8}$$

$$\epsilon_{\mu\nu\lambda\gamma}\eta^{(\pm)i}_{\gamma\sigma} = \mp\left(\delta_{\sigma\lambda}\eta^{(\pm)i}_{\mu\nu} + \delta_{\sigma\mu}\eta^{(\pm)i}_{\nu\lambda} - \delta_{\sigma\nu}\eta^{(\pm)i}_{\mu\lambda}\right) \tag{6.1.9}$$

$$\therefore \quad \epsilon^{ijk}\eta^{(\pm)j}_{\mu\nu}\eta^{(\pm)k}_{\rho\sigma} = \delta_{\mu\sigma}\eta^{(\pm)i}_{\rho\nu} - \delta_{\nu\sigma}\eta^{(\pm)i}_{\rho\mu} + \delta_{\rho\mu}\eta^{(\pm)i}_{\nu\sigma} - \delta_{\rho\nu}\eta^{(\pm)i}_{\mu\sigma} \tag{6.1.10}$$

6.2 Basic Killing tensors from Holten's Algorithm

6.2.1 Angular Momentum

If we choose to set $C^{(n)}_{\{i\}} = 0, \quad \forall \quad n \geq 2$, we get the Killing equations:

$$\nabla_{(i} C^{(1)}_{j)} = 0 \tag{6.2.1.1}$$

There are two parts of this solution we shall study in detail. We can write (6.2.1.1) as:

$$\nabla_i C^{(1)}_j + \nabla_j C^{(1)}_i = 0 \qquad \Rightarrow \qquad \nabla_i C^{(1)}_j = -\nabla_j C^{(1)}_i \tag{6.2.1.2}$$

This is an anti-symmetric matrix, written as $\theta_{ij} = -\theta_{ji}$. Further elaboration gives:

$$\theta_{ij}(\vec{x}) = \varepsilon_{ijk}(\vec{x})\theta^k = g_{im}(\vec{x})\varepsilon^m{}_{jk}\theta^k$$
$$\therefore \quad -\nabla_j C^{(1)}_i = g_{im}(\vec{x})\varepsilon^m{}_{jk}\theta^k \quad \Rightarrow \quad C^{(1)}_i = -g_{im}(\vec{x})\varepsilon^m{}_{jk}\theta^k x^j$$

Thus, we have the rotation operator as the 1st order co-efficient:

$$C^{(1)}_i = -g_{im}(\vec{x})\varepsilon^m{}_{jk}\theta^k x^j \tag{6.2.1.3}$$

Applying this co-efficient into the 1st term of the power series, we get:

$$Q^{(1)} = C^{(1)}_i \Pi^i = -g_{im}(\vec{x})\varepsilon^m{}_{jk}\theta^k x^j \Pi^i$$
$$\Rightarrow \qquad \boldsymbol{L}.\boldsymbol{\theta} = -\left(\varepsilon_{ijk}\Pi^i x^j\right)\theta^k = \left(\boldsymbol{x} \times \boldsymbol{\Pi}\right).\boldsymbol{\theta}$$
$$\therefore \qquad \boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{\Pi} \tag{6.2.1.4}$$

This eventually becomes the conserved quantity known as the angular momentum.

6.2.2 Laplace-Runge-Lenz vector

Now when we choose to set $C^{(n)}_{\{i\}} = 0, \quad \forall \quad n \geq 3$, we get the Killing equations:

$$\nabla_i C^{(2)}_{jk} + \nabla_j C^{(2)}_{ki} + \nabla_k C^{(2)}_{ij} = 0 \tag{6.2.2.1}$$

as we can see, (6.2.2.1) perfectly matches the property of the Killing Yano and Killing Stäckel tensors. The Runge-Lenz like quantity is given by a symmetric sum as shown below:

$$\left[\vec{A} \times \left(\vec{B} \times \vec{C}\right)\right]_i = \varepsilon_{ilm}\varepsilon^m{}_{jk} A^l B^j C^k \qquad \varepsilon_{ilm}\varepsilon^m{}_{jk} = \delta_{ij}\delta_{lk} - \delta_{ik}\delta_{lj}. \tag{6.2.2.2}$$

$$\begin{aligned}\nabla_k C^{(2)}_{ij} &= \varepsilon_{ilm}(\vec{x})\varepsilon^m{}_{jk}(\vec{x}) n^l + (i \leftrightarrow j) \\ &= \left(2g_{ij}(\vec{x})g_{kl}(\vec{x}) - g_{ik}(\vec{x})g_{jl}(\vec{x}) - g_{il}(\vec{x})g_{kj}(\vec{x})\right) n^l x^k,\end{aligned}$$

$$\therefore \quad C^{(2)}_{ij} = \left(2g_{ij}(\vec{x})n_k - g_{ik}(\vec{x})n_j - g_{kj}(\vec{x})n_i\right) x^k. \tag{6.2.2.3}$$

As before, applying this co-efficient to the 2nd order term in the power series gives

$$Q^{(2)} = \frac{1}{2} C^{(2)}_{ij} \Pi^i \Pi^j = \left\{ \left|\boldsymbol{\Pi}\right|^2 (\boldsymbol{n}.\boldsymbol{x}) - (\boldsymbol{\Pi}.\boldsymbol{x})(\boldsymbol{\Pi}.\boldsymbol{n}) \right\}$$
$$= \boldsymbol{N}.\boldsymbol{n} = \left\{ \left|\boldsymbol{\Pi}\right|^2 \boldsymbol{x} - (\boldsymbol{\Pi}.\boldsymbol{x})\boldsymbol{\Pi} \right\}.\boldsymbol{n} = \left\{\boldsymbol{\Pi} \times (\boldsymbol{x} \times \boldsymbol{\Pi})\right\}.\boldsymbol{n}$$
$$\therefore \qquad \boldsymbol{N} = \boldsymbol{\Pi} \times (\boldsymbol{x} \times \boldsymbol{\Pi}) \tag{6.2.2.4}$$

This quantity is a term that is present in another conserved quantity known as the Laplace-Runge-Lenz vector. Having found the two familiar types of conserved quantities, we can now proceed to see what it looks like for the Taub-NUT metric.

6.3 The Bohlin transformation

The Bohlin transformation that maps the co-ordinate system on a plane, is given by:

$$f : z \longrightarrow \xi^\alpha = (z^\alpha)^2 = Re^{i\phi} \quad \Rightarrow \quad z = \xi^{\frac{1}{2}} \tag{6.3.1}$$

Now we must note that another Noether invariant, the angular momentum will change form under this transformation. We re-parameterize to preserve the form of angular momentum.

$$l = r^2\dot{\theta} = |z|^2\dot{\theta} = |\xi|^2\phi' \quad \Rightarrow \quad |\xi|\frac{d\tilde{\tau}}{d\tau}\theta' = |\xi|^2\theta' \tag{6.3.2}$$

$$\therefore \quad \tau \longrightarrow \tilde{\tau} : \frac{d\tilde{\tau}}{d\tau} = |\xi| \tag{6.3.3}$$

The velocity and acceleration can be given as:

$$\dot{z}^\alpha = \frac{1}{2}\frac{|\xi|}{(\xi^\alpha)^{\frac{1}{2}}}\xi^{\alpha\prime} = \frac{1}{2}\left(\bar{\xi}^\alpha\right)^{\frac{1}{2}}\xi^{\alpha\prime} \tag{6.3.4}$$

$$\ddot{z}^\alpha = \frac{1}{2}|\xi|\frac{d}{d\tilde{\tau}}\left\{\left(\bar{\xi}^\alpha\right)^{\frac{1}{2}}\xi^{\alpha\prime}\right\} = \frac{1}{2}\frac{|\xi|^2}{(\xi^\alpha)^{\frac{1}{2}}}\xi^{\alpha\prime\prime} + \frac{1}{4}\left(\xi^\alpha\right)^{\frac{1}{2}}|\xi'|^2 \tag{6.3.5}$$

The equation of motion for a Harmonic Oscillator eventually becomes:

$$m\left\{\frac{1}{2}\frac{|\xi|^2}{(\xi^\alpha)^{\frac{1}{2}}}\xi^{\alpha\prime\prime} + \frac{1}{4}\left(\xi^\alpha\right)^{\frac{1}{2}}|\xi'|^2\right\} = -k\left(\xi^\alpha\right)^{\frac{1}{2}}$$

$$\Rightarrow \quad |\xi|^2\xi^{\alpha\prime\prime} + \frac{1}{2}\xi^\alpha|\xi'|^2 = -\frac{2k}{m}\xi^\alpha \quad \Rightarrow \quad \xi^{\alpha\prime\prime} = -\left(\frac{1}{2}|\xi'|^2 + \frac{2k}{m}\right)\frac{\xi^\alpha}{|\xi|^2} \tag{6.3.6}$$

The Hamiltonian $\mathcal{H}$ of the oscillator can be re-written to complete the transformation:

$$\mathcal{H} = \frac{m}{2}|\dot{z}|^2 + \frac{k}{2}|z|^2 = \frac{m}{4}\left(\frac{1}{2}|\xi'|^2 + \frac{2k}{m}\right)|\xi| \qquad \Rightarrow \qquad \left(\frac{|\xi'|^2}{2} + \frac{2k}{m}\right) = \frac{4\mathcal{H}}{m}\frac{1}{|\xi|} = \kappa\frac{1}{|\xi|}$$

$$\therefore \qquad \xi^{\alpha\prime\prime} = -\left(\frac{|\xi^{\alpha\prime}|^2}{2} + \frac{2k}{m}\right)\frac{\xi^\alpha}{|\xi|^2} \equiv -\kappa\frac{\xi^\alpha}{|\xi|^3} \tag{6.3.7}$$

showing that it restores the central force nature of the system, giving us the equation of motion for inverse square law forces.

6.4 Double derivative of Killing-Yano tensors

Similar to Killing vectors, rank n Killing-Yano tensors exhibit a curvature equation

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\, K_{c_1\dots c_n} = \sum_{i=1}^{n} R_{abc_i}{}^{d} K_{c_1\dots d\dots c_n}. \tag{6.4.1}$$

For the LHS of (3.2.3.3), by permuting the indices according to the rules, we will get

$$\begin{aligned}(\nabla_a\nabla_b - \nabla_b\nabla_a)\, K_{c_1\dots c_n} &= -\nabla_a\nabla_{c_1}K_{bc_2\dots c_n} + \nabla_b\nabla_{c_1}K_{ac_2\dots c_n}\\ &= 2\nabla_{c_1}\nabla_b K_{ac_2\dots c_n} - R_{abc_1}{}^{d}K_{dc_2\dots c_n}\\ &\qquad + \sum_{i=2}^{n}\left(R_{bc_1c_i}{}^{d}K_{ac_2\dots d\dots c_n} - R_{ac_1c_i}{}^{d}K_{bc_2\dots d\dots c_n}\right)\\ &= R_{abc_1}{}^{d}K_{dc_2\dots c_n} + \sum_{i=2}^{n} R_{abc_i}{}^{d}K_{c_1\dots d\dots c_n}\end{aligned}$$

$$\begin{aligned}2\nabla_{c_1}\nabla_b K_{ac_2\dots c_n} = 2R_{abc_1}{}^{d}K_{dc_2\dots c_n} + \Big(&\underbrace{R_{abc_i}{}^{d}K_{c_1\dots d\dots c_n}}_{\mathrm{I}}\\ &+ \sum_{i=2}^{n}\underbrace{R_{ac_1c_i}{}^{d}K_{bc_2\dots d\dots c_n} - R_{bc_1c_i}{}^{d}K_{ac_2\dots d\dots c_n}}_{\mathrm{II}}\Big)\end{aligned} \tag{6.4.2}$$

$$\begin{aligned}\because \qquad \nabla_{c_1}\nabla_b K_{ac_2\dots c_n} &= \nabla_{c_1}\nabla_{[b}K_{ac_2\dots c_n]},\\ W_{c_1} &:= \nabla_{c_1}\nabla_{[b}K_{ac_2\dots c_n]}e^a\wedge e^b \wedge e^{c_2}\dots\wedge e^{c_n}.\end{aligned}$$

On writing (6.4.2) as a 3-form, we can say that for I and II

$$\begin{aligned}\mathrm{I}\ :&\qquad R_{abc_i}{}^{d}e^a\wedge e^b\wedge e^{c_i} = \frac{1}{3}\left(R_{abc_i}{}^{d} + R_{bc_ia}{}^{d} + R_{c_iab}{}^{d}\right)e^a\wedge e^b\wedge e^{c_i} = 0\\ \mathrm{II}\ :&\qquad R_{ac_1c_i}{}^{d}K_{bc_2\dots d\dots c_n}e^a\wedge e^b\wedge e^{c_i} = -R_{c_1c_ib}{}^{d}K_{ac_2\dots d\dots c_n}e^a\wedge e^b\wedge e^{c_i}.\end{aligned}$$

Thus, on using Bianchi identity for curvature, II of (6.4.2) will become:

$$\begin{aligned}&-\sum_{i=2}^{n}\left(R_{c_1c_ib}{}^{d} + R_{bc_1c_i}{}^{d}\right)K_{ac_2\dots d\dots c_n}e^a\wedge e^b\wedge e^{c_i}\\ &= \sum_{i=2}^{n}R_{c_ibc_1}{}^{d}K_{ac_2\dots d\dots c_n}e^a\wedge e^b\wedge e^{c_i}\\ &= \sum_{i=2}^{n}R_{abc_1}{}^{d}K_{dc_2\dots c_i\dots c_n}e^a\wedge e^b\wedge e^{c_i} = (n-1)R_{abc_1}{}^{d}K_{dc_2\dots c_i\dots c_n}e^a\wedge e^b\wedge e^{c_i}.\end{aligned}$$

Applying this result back in the main equation (6.4.2), we get:

$$2\nabla_{c_1}\nabla_b K_{ac_2\dots c_n}e^a\wedge e^b\wedge e^{c_i} = \left[2R_{abc_1}{}^{d} + (n-1)R_{abc_1}{}^{d}\right]K_{dc_2\dots c_n}e^a\wedge e^b\wedge e^{c_i}$$

$$2\nabla_{c_1}\nabla_b K_{ac_2\dots c_n}e^a\wedge e^b\wedge e^{c_i} = (n+1)R_{abc_1}{}^{d}K_{dc_2\dots c_n}e^a\wedge e^b\wedge e^{c_i}.$$

Finally, we get the double-derivative of KY tensors as:

$$\therefore \qquad \nabla_a\nabla_b K_{c_1c_2\dots c_n} = (-1)^{n+1}\frac{n+1}{2}R_{[bc_1|a|}{}^{d}K_{c_2c_3\dots c_n]d}. \tag{6.4.3}$$

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